●●●●○ Level 4 · Mathematics M4.4

Similarity, transformations and circles

Move, turn, flip and enlarge shapes, and learn the angle rules hidden in circles.

Reflect, rotate, translate and enlarge shapes on a grid and describe each move exactly. Prove triangles congruent, use similar shapes to find missing lengths, and see how areas and volumes grow with the scale factor. Then find arc lengths and sector areas, and use the angle rules of the circle.

YOU ARE READY FOR THIS BOOK IF

You can reflect a shape in the y axis on a grid.

One step up	M5.3 Trigonometric functions
This book	**M4.4 Similarity, transformations and circles**
One step down	M3.5 Angles, triangles and symmetry

Contents

M4.4.1 Reflections and rotations

M4.4.2 Translations and combined moves

M4.4.3 Enlargements

M4.4.4 Congruent triangles

M4.4.5 Similar shapes

M4.4.6 Similar areas and volumes

M4.4.7 Arcs and sectors

M4.4.8 Angle rules in circles

Key words, A to Z

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M4.4.1 CHAPTER 1

Reflections and rotations

I flip a shape over the line y = x. Where does the corner at (1, 3) go?

YOUR GUESS

Pick one before you read on. It is fine to be wrong.

- ☐ **A** To (1, −3). It is the same distance below the x axis.
- ☐ **B** To (3, 1). The two numbers swap.
- ☐ **C** To (−1, 3). It is on the other side of the y axis.

IF THIS FEELS HARD, READ THIS FIRST

- Level 3, Angles, triangles and symmetry, M3.5.6. If a mirror line and a reflection look new, read this first.
- Level 3, Coordinates and straight lines, M3.9.3. If reflecting in the x axis or the y axis looks new, read this first.
- Level 3, Angles, triangles and symmetry, M3.5.7. If turning a shape about a point looks new, read this first.

BY THE END YOU CAN

- Reflect a shape in a line like y = x
- Rotate a shape by 90° or 180° about a point
- Describe a rotation by its centre, angle and direction

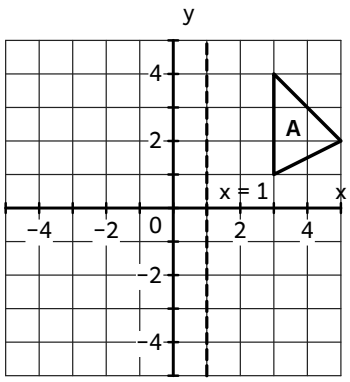
A shape on a grid can be flipped, turned, slid or made bigger. Each of these is a **transformation**. This chapter has the flip and the turn. The next two chapters have the slide and the enlargement.

A **reflection** flips a shape over a **mirror line**. The new shape is called the **image**. Each point and its image are the same distance from the mirror line, on opposite

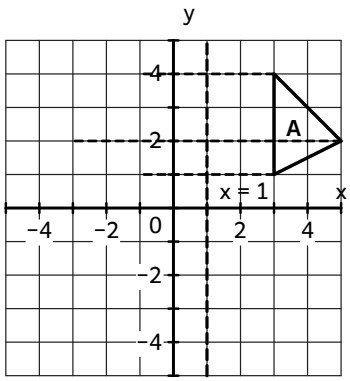
sides. The line joining them meets the mirror line at a right angle.

DRAW IT FIRST

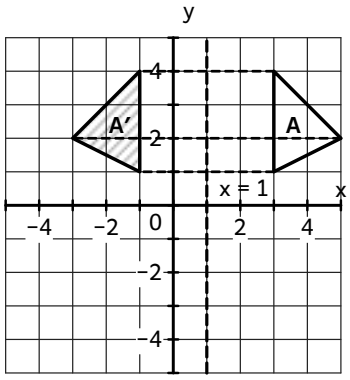
Reflect triangle A in the line x = 1



1 Triangle A has corners at (3, 1), (5, 2) and (3, 4). Draw the mirror line x = 1. Every point on it has x equal to 1.



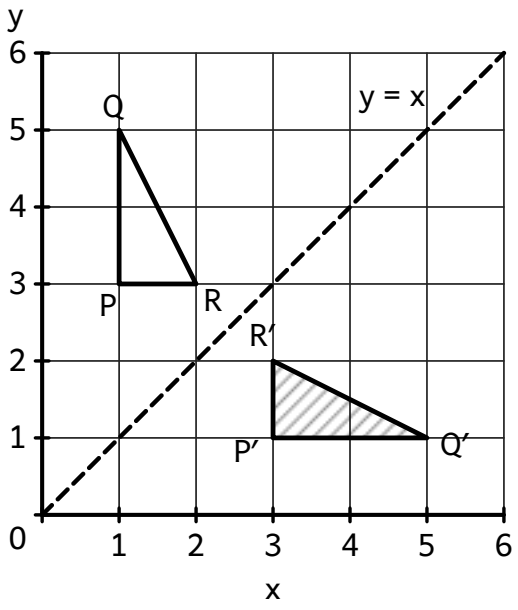
2 From each corner, draw a line straight across the mirror. Count how far the corner is from the mirror.



3 Go the same distance past the mirror. Join the new points to make the image, A′.

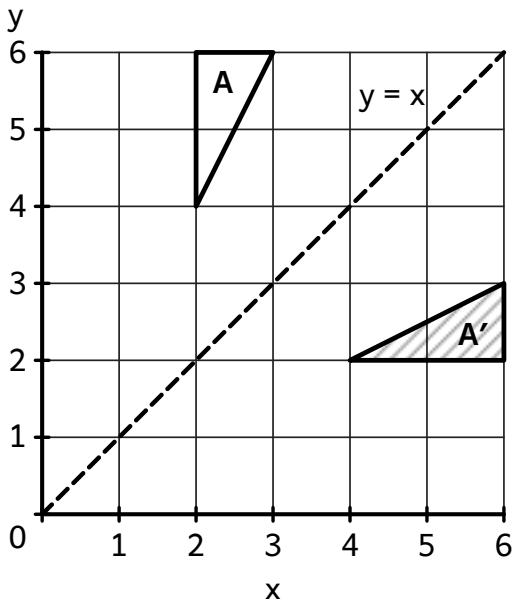
The corner (3, 1) is 2 units from the line x = 1. So its image is 2 units on the other side, at (−1, 1). A point that lies on the mirror line is its own image.

Now take a mirror line that slants. The line y = x passes through (1, 1), (2, 2) and (3, 3). Every point on it has equal x and y.



The mirror line y = x. Triangle PQR and its image lie on opposite sides.

The line y = x makes the same angle with both axes. Folding along it swaps the two axes, so a distance across becomes a distance up. A point (a, b) goes to the point (b, a). The numbers swap.



The triangle with corners (2, 4), (2, 6) and (3, 6), and its image in y = x.

WORKED EXAMPLE

Reflect the triangle with corners (2, 4), (2, 6) and (3, 6) in the line y = x.

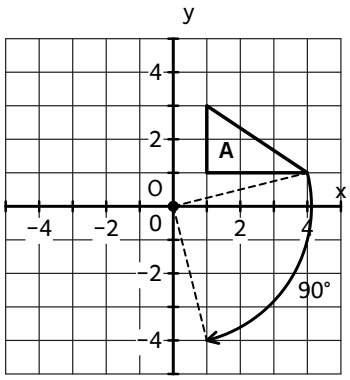
1. Swap the two numbers of each corner.
2. The corner (2, 4) goes to (4, 2).
3. The corner (2, 6) goes to (6, 2).
4. The corner (3, 6) goes to (6, 3).

Answer: (4, 2), (6, 2) and (6, 3)

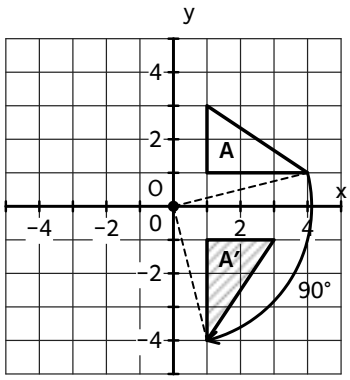
A **rotation** turns a shape about one fixed point. That point is the **centre of rotation**. A turn can be **clockwise**, the way the hands of a clock move, or **anticlockwise**, the other way. A half turn of 180° needs no direction, because both ways end in the same place.

DRAW IT FIRST

Turn triangle A by 90° clockwise about O

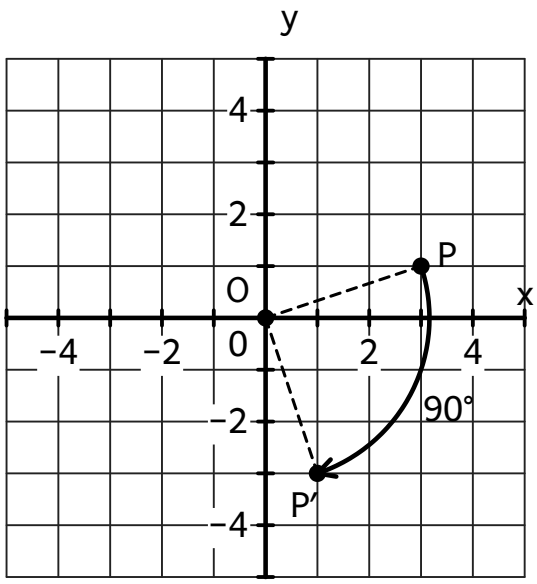


1 Triangle A has corners at (1, 1), (4, 1) and (1, 3). The centre of rotation is O at (0, 0). Take the corner (4, 1). Join it to O, then turn that line 90° clockwise. The corner lands at (1, −4).



2 Turn every corner the same way. Join the new corners to make the image, A′.

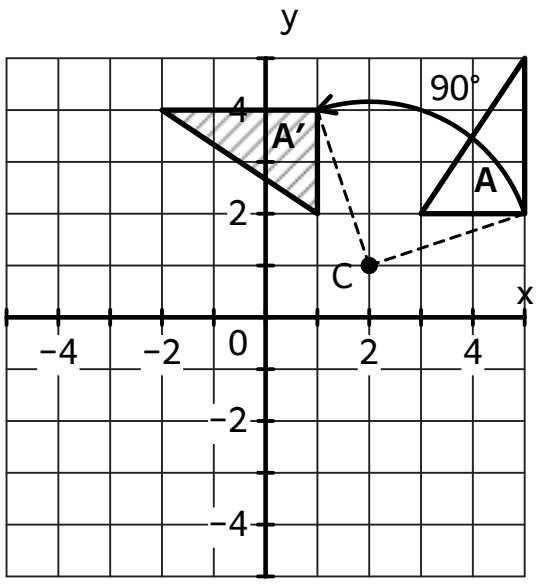
A turn keeps every point the same distance from the centre. The shape does not change size and does not flip over.



P is 3 right and 1 up from O. After a 90° clockwise turn, P′ is 1 right and 3 down.

Look at P. Going 3 right and 1 up from O becomes going 1 right and 3 down. The right step turned into a down step, and the up step turned into a right step. So a 90° clockwise turn about O sends (x, y) to (y, −x).

A 90° anticlockwise turn about O sends (x, y) to (−y, x). A half turn of 180° about O sends (x, y) to (−x, −y). Each is the same step, turned by the angle.



Triangle A turned 90° anticlockwise about C at (2, 1).

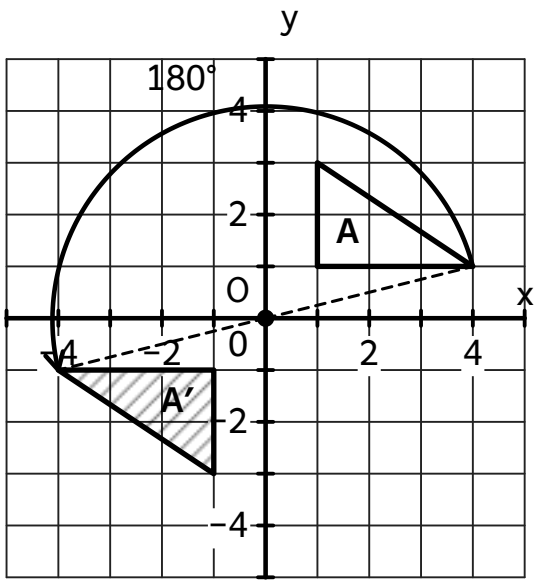
WORKED EXAMPLE

Turn the triangle with corners (3, 2), (5, 2) and (5, 5) by 90° anticlockwise about C at (2, 1).

1. Find how far each corner is from C. The corner (3, 2) is 1 right and 1 up. The corner (5, 2) is 3 right and 1 up. The corner (5, 5) is 3 right and 4 up.
2. A 90° anticlockwise turn changes "right r, up u" into "left u, up r". So (3, 2) is now 1 left and 1 up from C.
3. The corner (3, 2) goes to (1, 2). The corner (5, 2) goes to (1, 4). The corner (5, 5) goes to (−2, 4).

Answer: (1, 2), (1, 4) and (−2, 4)

To describe a rotation, say all three things: the centre, the angle and the direction. For a half turn, the direction can be left out. "90° clockwise about (0, 0)" is complete. "90° clockwise" is not.



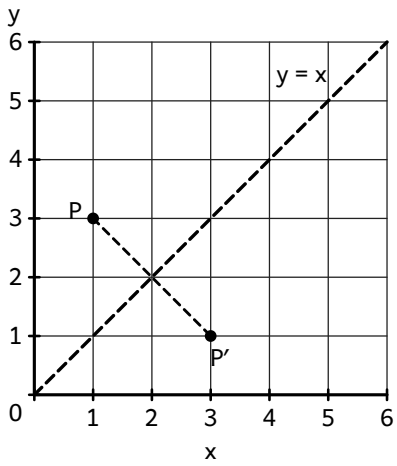
A half turn about O. The same image comes from turning clockwise or anticlockwise.

SEE IT THREE WAYS

With things

Fold a square sheet along its diagonal. Hold it to the light. A dot on one half shows through on the other half, exactly where its reflection is.

As a picture



In symbols

A reflection in y = x sends (*a*, b) to (b, *a*). A 90° clockwise turn about O sends (x, y) to (y, −x).

TRY IT

Fold along y = x

You need: a square sheet of thin paper, a pencil, a ruler.

1. Draw an x axis and a y axis along two edges of the paper, with the origin in one corner. Mark every 1 cm.
2. Draw the line from the origin to the opposite corner. That is the line y = x.
3. Mark the point (1, 3) with a dot. Fold the paper along y = x, so the two halves lie on top of each other.
4. Hold the folded sheet up to the light. Put a second dot on the top half, exactly over the dot that shows through.

What you will see: Open the sheet. The second dot is at (3, 1). The fold swapped the two numbers.

If you cannot try it, this is what happens.

Dot	Before	After
P	(1, 3)	(3, 1)

A WRONG IDEA MANY PEOPLE HAVE

~~A 90° turn about (0, 0) gives the same picture as a reflection in y = x.~~

Why it feels right: Both moves keep the size of the shape, and both put the numbers of a point in a new order.

The test that breaks it: Take the corner (1, 3). Turn it 90° clockwise about (0, 0) and it goes to (3, −1). Reflect it in y = x and it goes to (3, 1). These are different points.

What is true: A reflection flips the shape over, so its corners run round the other way. A rotation does not flip it. They are different transformations.

KEY WORDS

transformation
a rule that moves a shape: a flip, a turn, a slide or an enlargement.

reflection
a flip of a shape over a mirror line.

mirror line
the line a shape is flipped over. A point and its image are the same distance from it.

image
the shape that a transformation makes.

rotation
a turn of a shape about a fixed point.

centre of rotation
the point that stays still while the shape turns.

clockwise
in the direction the hands of a clock move.

anticlockwise
in the opposite direction to the hands of a clock.

BACK TO YOUR GUESS

The answer is B: To (3, 1). The two numbers swap.

The line y = x runs through every point with equal x and y, such as (2, 2). A flip over it swaps the two numbers, so (1, 3) goes to (3, 1). The other two answers are flips over the x axis and over the y axis.

CHECK YOURSELF

Try each one before you look. A wrong answer tells you where to go next.

1. Triangle A has a corner at (2, 5). A is reflected in the line y = x. Where is that corner now?

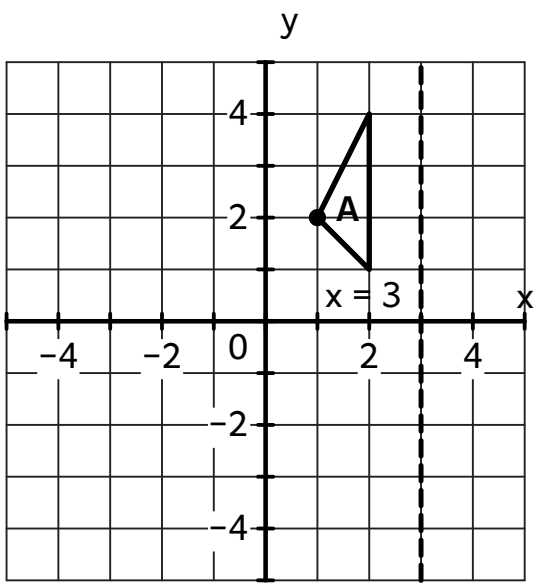
   A (−2, 5)

   B (5, 2)

   C (2, −5)

   Answer at the back of the book.

2. Triangle A is reflected in the mirror line x = 3. Its corner at (1, 2) is marked. Where is the image of that corner?



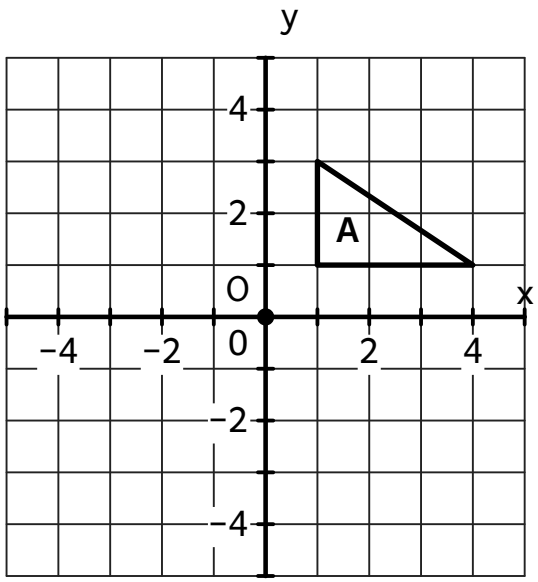
   A (−1, 2)

   B (3, 2)

   C (5, 2)

   Answer at the back of the book.

3. Triangle A is turned 90° clockwise about O at (0, 0). Where does the corner (4, 1) go?



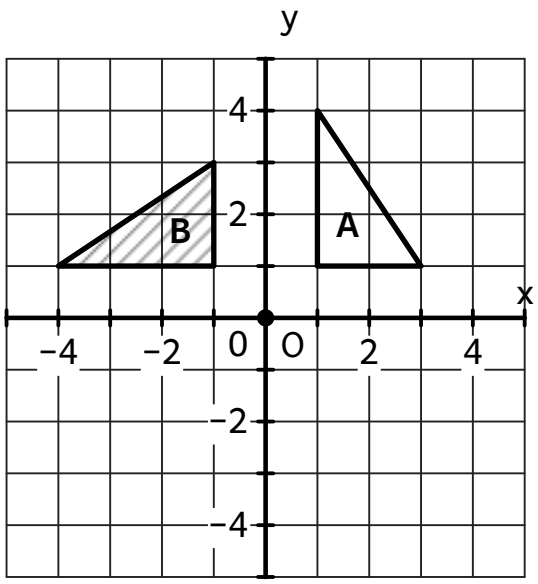
A (1, −4)

B (−1, 4)

C (−4, −1)

Answer at the back of the book.

4. Triangle A is moved to triangle B by a rotation. Which rotation is it?



A 90° clockwise about O

B 180° about O

C 90° anticlockwise about O

D 90° anticlockwise about (1, 1)

Answer at the back of the book.

5. Which description of a rotation is complete?

A 90° clockwise

B a turn about (2, 1)

C 90° about (2, 1)

D 90° clockwise about (2, 1)

Answer at the back of the book.

6. A point is reflected in the line y = x. It starts at (7, 2). What is the x number of the image?

Answer at the back of the book.

7. A corner at (1, 4) is reflected in the line x = 3. What is the x number of the image?

Answer at the back of the book.

8. A point is 4 units right of the centre C and 6 units above it. It makes a half turn about C. How many units below C is the image?

   Answer at the back of the book.

9. A point at (2, 6) is turned 90° clockwise about O at (0, 0). How many units below the x axis is the image?

   Answer at the back of the book.

KEEP IT

Copy this into your exercise book.

A reflection flips a shape over a mirror line. In y = x the two numbers swap. A rotation needs a centre, an angle and a direction. A half turn needs no direction.

Draw: A triangle and its image in the line y = x, with the corners named.

M4.4.2 CHAPTER 2

Translations and combined moves

A shape slides 3 squares right and 2 squares up. How do I write that move with numbers?

YOUR GUESS

Pick one before you read on. It is fine to be wrong.

- ☐ **A** With one number, 5, because the shape moves 3 + 2 squares.
- ☐ **B** With two numbers in a column: 3 on top and 2 below.
- ☐ **C** With two numbers in a column: 2 on top and 3 below.

IF THIS FEELS HARD, READ THIS FIRST

- Level 3, Coordinates and straight lines, M3.9.3. If moving a shape on a grid looks new, read this first.
- Level 3, Coordinates and straight lines, M3.9.2. If points with negative numbers look new, read this first.

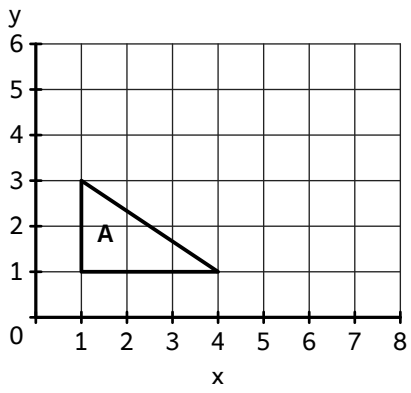
BY THE END YOU CAN

- Translate a shape by a column vector
- Describe a translation as a column vector
- Find one move that does the same as two moves

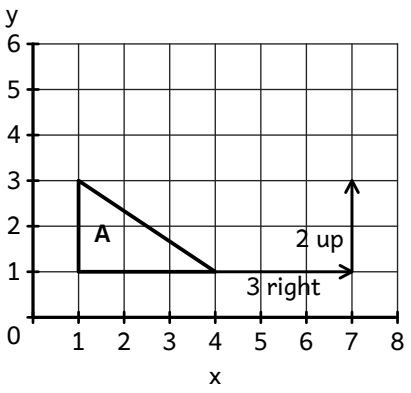
A **translation** slides a shape. It does not turn the shape, flip it or change its size. Every point of the shape moves the same distance in the same direction. So the image faces the same way as the shape.

DRAW IT FIRST

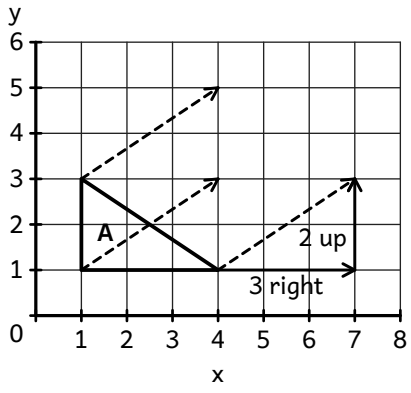
Slide triangle A 3 right and 2 up



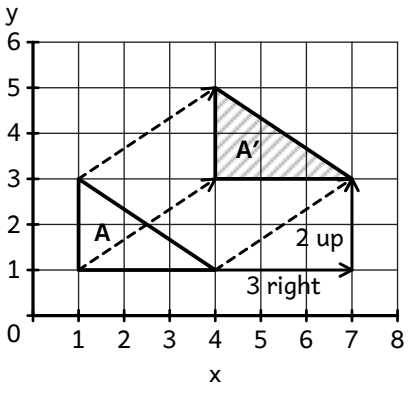
1 Triangle A has corners at (1, 1), (4, 1) and (1, 3).



2 Take one corner. Move it 3 squares right, then 2 squares up.



3 Move every other corner the same way. The three arrows are parallel and equally long.



4 Join the new corners. This is the image, A′.

We write a slide as a **column vector**: two numbers in brackets, one above the other. The top number is the move right. The bottom number is the move up. A minus sign means left, for the top number, or down, for the bottom number.

$$\begin{bmatrix} 3 \\ 2 \end{bmatrix} \quad \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

3 right, 2 up 2 left, 1 up

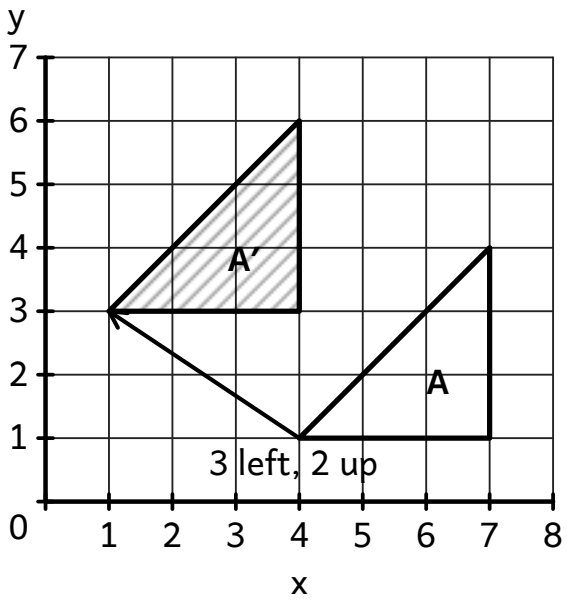
The top number is the move right. The bottom number is the move up.

$$\begin{bmatrix} 0 \\ -4 \end{bmatrix} \quad \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

4 down 1 right, 1 down

A zero means no move in that direction. A minus sign means left or down.

To translate a point by a vector, add the top number to x and the bottom number to y. A vector says how far to move. It does not say where to land.



Triangle A and its image after a slide of 3 left and 2 up.

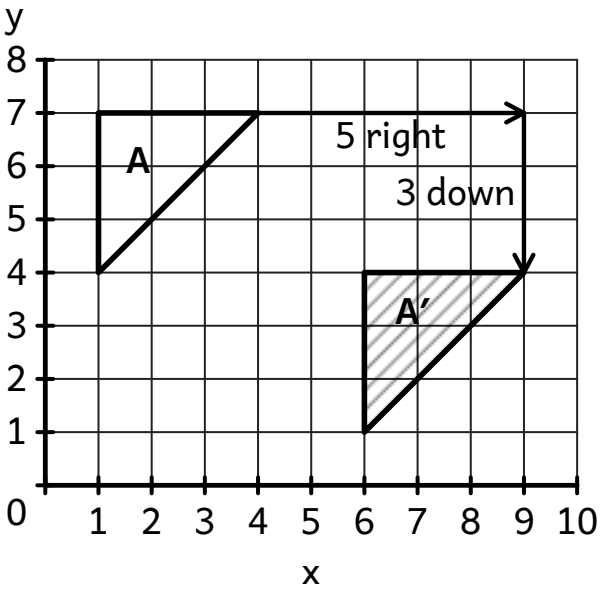
WORKED EXAMPLE

Translate the triangle with corners (4, 1), (7, 1) and (7, 4) by the vector with −3 on top and 2 below.

1. The top number is −3, so move every corner 3 left. The bottom number is 2, so move every corner 2 up.
2. The corner (4, 1) goes to (1, 3).
3. The corner (7, 1) goes to (4, 3).
4. The corner (7, 4) goes to (4, 6).

Answer: (1, 3), (4, 3) and (4, 6)

Now go the other way. Pick one corner of the shape and its image. The move right is the new x minus the old x. The move up is the new y minus the old y.



The corner (4, 7) moves to (9, 4).

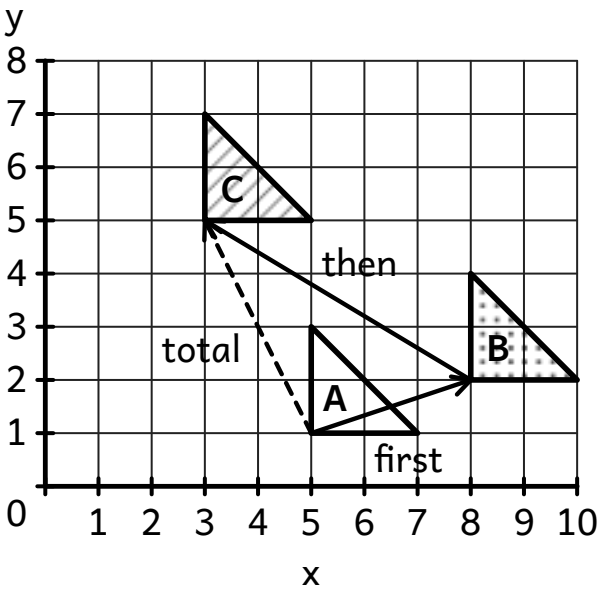
WORKED EXAMPLE

The corner (4, 7) of triangle A moves to (9, 4) in the image. Write the translation as a column vector.

1. Pick one corner and its image. The corner is (4, 7) and its image is (9, 4).
2. The move right is the new x minus the old x: from 4 to 9 is 5.
3. The move up is the new y minus the old y: from 7 to 4 is −3. That is 3 down.
4. The column vector has 5 on top and −3 below.

Answer: 5 on top and −3 below

Two slides in a row make one slide. Add the top numbers and add the bottom numbers. Two moves one after the other are a **<u>combined move</u>**. Take the vector with 3 on top and 1 below, then the vector with −5 on top and 3 below. Together they make one slide of 2 left and 4 up.



A moves to B, then B moves to C. One slide takes A straight to C.

WORKED EXAMPLE

A shape moves by the vector with 3 on top and 1 below. Then it moves by the vector with −5 on top and 3 below. Find one translation that does the same.

1. Add the top numbers: 3 and −5 make −2.
2. Add the bottom numbers: 1 and 3 make 4.
3. The one move has −2 on top and 4 below.
4. Check with the corner (5, 1). The first move takes it to (8, 2).
5. The second move takes (8, 2) to (3, 5). The one move takes (5, 1) to (3, 5) as well.

Answer: −2 on top and 4 below

SEE IT THREE WAYS

With things

Slide a book across a table: 3 hand spans to the right and 2 hand spans away from you. Every corner of the book moves the same way.

As a picture

$$\begin{bmatrix} 3 \\ 2 \end{bmatrix}$$

3 right, 2 up

In symbols

The vector with 3 on top and 2 below moves the point (x, y) to (x + 3, y + 2).

TRY IT

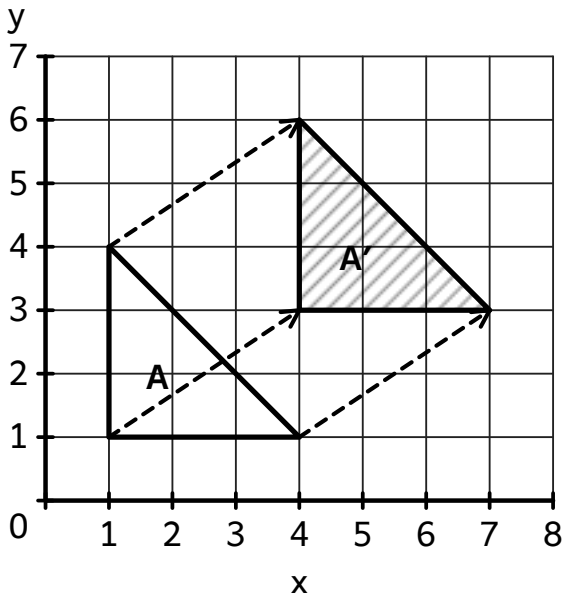
Slide *a* tracing

You need: squared paper, thin paper to trace on, *a* pencil.

1. Draw *a* small triangle on the squared paper. Trace it on the thin paper.
2. Slide the tracing 3 squares right and 2 squares up. Keep its edges parallel to the grid lines. Do not turn it.
3. Press through, or draw round the tracing, to mark the image.
4. Join each corner of the triangle to the matching corner of the image.

What you will see: The three lines are parallel and have the same length. Every corner moved by the same vector.

If you cannot try it, this is what happens.



A WRONG IDEA MANY PEOPLE HAVE

~~The vector with 3 on top and 2 below moves a shape to the point (3, 2).~~

Why it feels right: It has the same two numbers as the point, and a point is written with the x number first.

The test that breaks it: Take the corner (1, 1) and the vector with 3 on top and 2 below. Add 3 to x and 2 to y. The corner goes to (4, 3), not to (3, 2).

What is true: A column vector says how far to move: right, then up. It does not say where to land. To find the image, add the vector to each corner.

KEY WORDS

translation
a slide of a shape. Every point moves the same distance in the same direction.

column vector
two numbers in brackets, one above the other. The top number is the move right and the bottom number is the move up.

combined move
two moves one after the other. For two translations it is one translation.

BACK TO YOUR GUESS

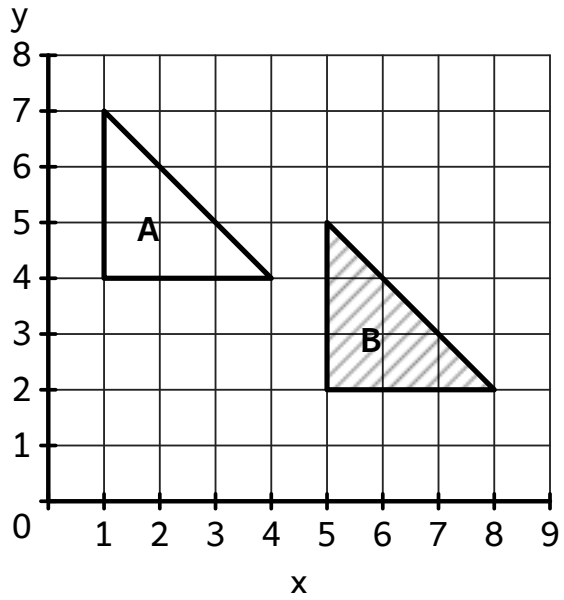
The answer is B: With two numbers in a column: 3 on top and 2 below.

A slide has two parts, a move right and a move up, so it needs two numbers. By agreement the move right goes on top and the move up goes below. One number cannot tell the two parts apart.

CHECK YOURSELF

Try each one before you look. A wrong answer tells you where to go next.

1. Triangle A moves to triangle B by a translation. Which column vector describes the move?



A top number −2, bottom number 4

B top number 4, bottom number −2

C top number −4, bottom number 2

D top number 4, bottom number 2

Answer at the back of the book.

2. The corner (2, −1) is translated by the vector shown. Where does the corner go?

$$\begin{bmatrix} 4 \\ 3 \end{bmatrix}$$

v

A (5, 3)

B (4, 3)

C (6, 2)

D (−2, −4)

Answer at the back of the book.

3. A shape moves by the vector with 2 on top and 1 below. Then it moves by the vector with −5 on top and 2 below. Which one vector does the same?

   A −3 on top, 3 below

   B 7 on top, −1 below

   C −10 on top, 2 below

   Answer at the back of the book.

4. A shape moves 5 units left and does not move up or down. Which column vector is that?

   A 0 on top, 5 below

   B 5 on top, 0 below

   C −5 on top, 0 below

   Answer at the back of the book.

5. A translation moves the point (1, 1) to (4, −1). Where does the same translation move the point (0, 0)?

   A (4, −1)

   B (−3, 2)

   C (0, 0)

   D (3, −2)

   Answer at the back of the book.

6. **From an earlier chapter**

   The point (4, 1) is turned 90° anticlockwise about (0, 0). What is the y number of the image?

   Answer at the back of the book.

7. The corner (3, 2) is translated by the vector with 4 on top and 5 below. What is the y number of the image?

   Answer at the back of the book.

8. A corner moves from (1, 2) to (6, 9). What is the number on the bottom of the column vector?

   Answer at the back of the book.

9. A shape moves by the vector with 2 on top and 5 below. Then it moves by the vector with −6 on top and 4 below. What is the number on the bottom of the one vector that does the same?

   Answer at the back of the book.

KEEP IT

Copy this into your exercise book.

A translation slides a shape. The column vector has the move right on top and the move up below. To combine two moves, add the tops and add the bottoms.

Draw: A triangle and its image with an arrow from one corner, and the vector written beside it.

Enlargements

> **A shape is enlarged by scale factor 2 about a point. Where does a corner that is 3 squares right of the point go?**

YOUR GUESS

Pick one before you read on. It is fine to be wrong.

- ☐ **A** 6 squares right of the point. Every distance from the point doubles.
- ☐ **B** 5 squares right of the point. The corner moves 2 squares further.
- ☐ **C** 3 squares right of the point. A corner stays where it is.

IF THIS FEELS HARD, READ THIS FIRST

- Level 3, Ratio and proportion, M3.4.7. If a scale such as 1 : 50 000 looks new, read this first.
- Level 3, Fractions, decimals and percentages, M3.3.3. If finding a half of a number looks new, read this first.
- Level 3, Coordinates and straight lines, M3.9.3. If points on a grid look new, read this first.

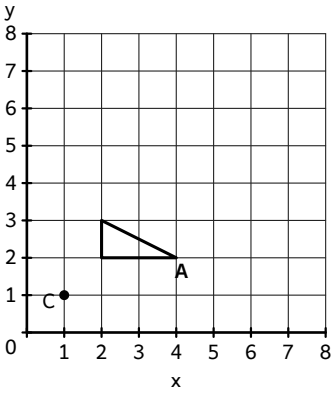
BY THE END YOU CAN

- Enlarge a shape by a scale factor from a centre
- Enlarge a shape by a scale factor of 1/2
- Find the centre of an enlargement

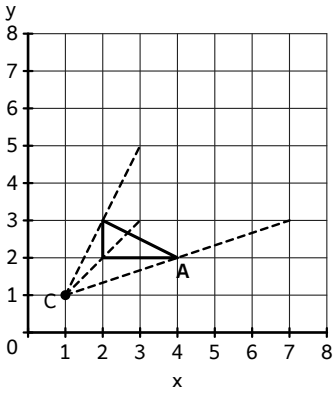
An **enlargement** changes the size of a shape and keeps its form. To describe one you need two things. The first is a point called the **centre of enlargement**. The second is a number called the **scale factor**. Every distance from the centre is multiplied by the scale factor.

DRAW IT FIRST

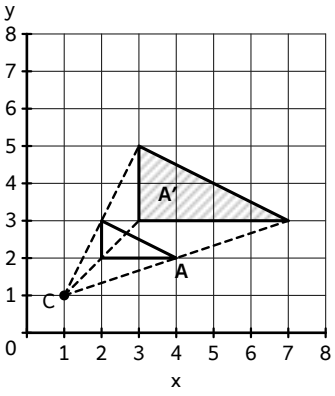
Enlarge triangle A by scale factor 2 about C



1 Triangle A has corners at (2, 2), (4, 2) and (2, 3). The centre of enlargement is C at (1, 1).



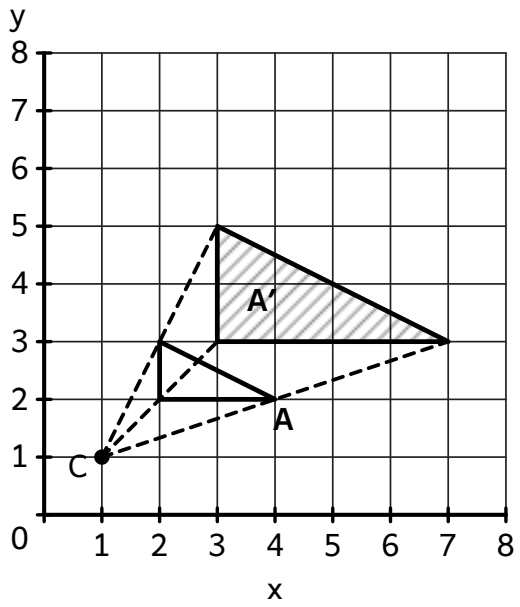
2 Draw a line from C through each corner of A. Go on to twice the distance from C. Mark each end.



3 Join the three marks. This is the image, A′. Each side is twice as long as the side it matches.

Look at the corner (2, 2). It is 1 right and 1 up from C. Its image is 2 right and 2 up from C, which is the point (3, 3). The rule is: image = C + scale factor × (corner − C).

An enlargement changes lengths and keeps angles. Each side of A′ is twice the side it matches. Every angle of A′ equals the angle it matches. That is why the shape keeps its form.



The image corners lie on the lines from C, at twice the distance.

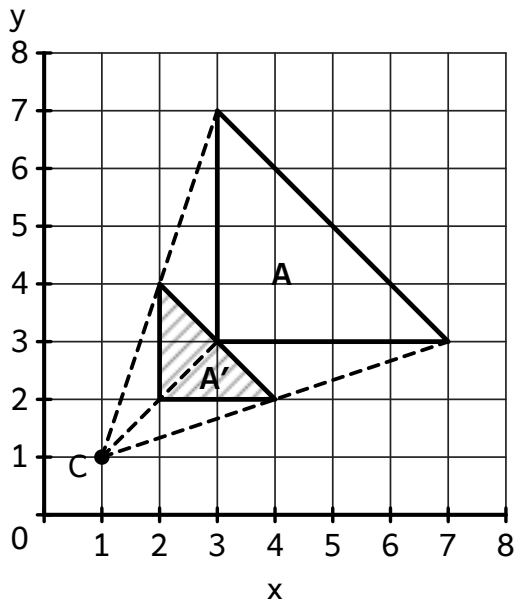
WORKED EXAMPLE

Enlarge the triangle with corners (2, 2), (4, 2) and (2, 3) by scale factor 2 about C at (1, 1).

1. Find each corner as a step from C. The corner (2, 2) is 1 right and 1 up. The corner (4, 2) is 3 right and 1 up. The corner (2, 3) is 1 right and 2 up.
2. Multiply each step by 2. The steps become 2 right and 2 up, then 6 right and 2 up, then 2 right and 4 up.
3. Count from C at (1, 1). The image corners are (3, 3), (7, 3) and (3, 5).

Answer: (3, 3), (7, 3) and (3, 5)

A scale factor between 0 and 1 makes the shape smaller. With scale factor $^1/_2$ every distance from the centre is halved. The image is half as long and lies between the centre and the original.



Scale factor 1/2 about C. Triangle A′ is half the size of A.

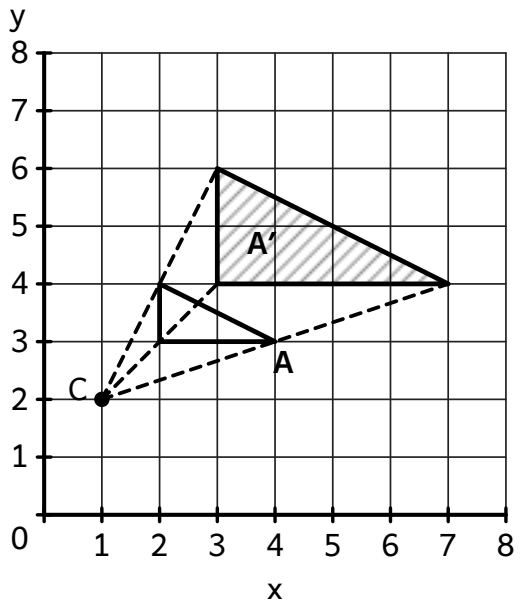
WORKED EXAMPLE

Enlarge the triangle with corners (3, 3), (7, 3) and (3, 7) by scale factor $^1/_2$ about C at (1, 1).

1. Find each corner as a step from C. The corner (3, 3) is 2 right and 2 up. The corner (7, 3) is 6 right and 2 up. The corner (3, 7) is 2 right and 6 up.
2. Halve each step. The steps become 1 right and 1 up, then 3 right and 1 up, then 1 right and 3 up.
3. Count from C at (1, 1). The image corners are (2, 2), (4, 2) and (2, 4). Each side is half as long.

Answer: (2, 2), (4, 2) and (2, 4)

Now suppose you are given a shape and its image, and you need the centre. The line through a corner and its image passes through the centre. Draw two such lines. They meet at the centre.



The lines through matching corners all meet at C.

WORKED EXAMPLE

Triangle A has corners (2, 3), (4, 3) and (2, 4). Its image has corners (3, 4), (7, 4) and (3, 6). Find the centre and the scale factor.

1. Join the corner (2, 3) to its image (3, 4). Draw the line on past (2, 3).
2. Do the same for the corner (4, 3) and its image (7, 4).
3. The two lines cross at (1, 2). That point is the centre.
4. From C the corner (2, 3) is 1 across. The image (3, 4) is 2 across. The scale factor is 2.

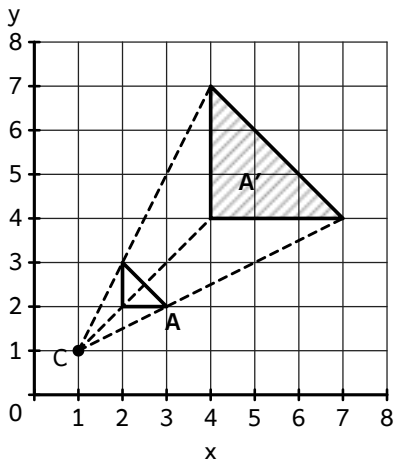
Answer: centre (1, 2), scale factor 2

SEE IT THREE WAYS

With things

Draw a small flag on squared paper. Draw it again on paper with squares 3 times as wide. Every length is 3 times as long.

As a picture



In symbols

With scale factor 3 about C at (1, 1), the corner (3, 2) goes to (7, 4). Every length is 3 times as long.

TRY IT

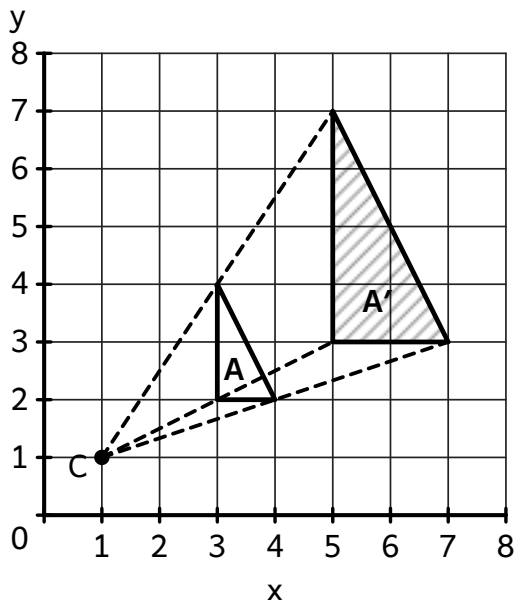
Double *a* triangle with *a* ruler

You need: squared paper, *a* ruler, *a* pencil.

1. Mark *a* point C near the bottom left of the paper. Draw *a* small triangle some way from C.
2. Draw *a* line from C through each corner of the triangle. Make each line long.
3. Measure from C to *a* corner. Mark *a* point on the line that is twice that distance from C. Do the same for each corner.
4. Join the three new points. Measure one side of each triangle.

What you will see: Each side of the new triangle is twice the side it matches. Check the angles with *a* corner of the paper: they look the same.

If you cannot try it, this is what happens.



A WRONG IDEA MANY PEOPLE HAVE

~~To enlarge a shape by scale factor 2, add 2 to every length.~~

Why it feels right: People say "make it 2 bigger", and adding is the first way to make a number bigger.

The test that breaks it: Take a triangle with sides 3 cm and 6 cm. Add 2 to each and they become 5 cm and 8 cm. The first shape had a 1 : 2 ratio of sides. The new one does not, so it is a different shape.

What is true: An enlargement multiplies every length by the scale factor. The sides 3 cm and 6 cm become 6 cm and 12 cm. They are still in the ratio 1 : 2.

KEY WORDS

enlargement
a transformation that multiplies every distance from a centre by the same number. The shape keeps its form.

centre of enlargement
the point the distances are measured from. It does not move.

scale factor
the number every length is multiplied by. A scale factor between 0 and 1 makes the shape smaller.

BACK TO YOUR GUESS

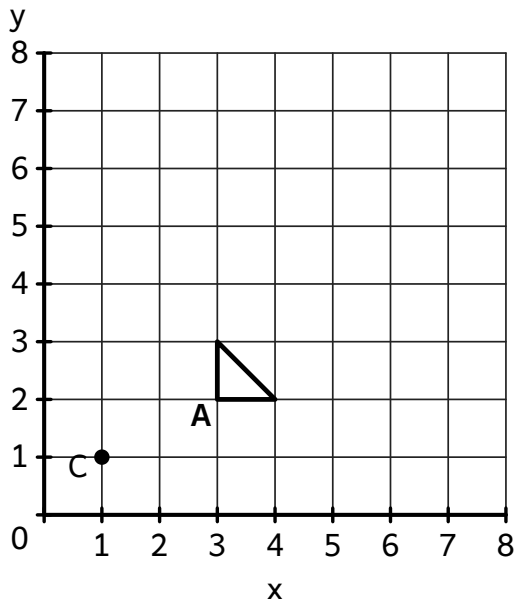
The answer is A: 6 squares right of the point. Every distance from the point doubles.

An enlargement multiplies every distance from the centre by the scale factor. A distance of 3 squares becomes 2 × 3 = 6 squares. Adding 2 would not keep the shape the same.

CHECK YOURSELF

Try each one before you look. A wrong answer tells you where to go next.

1. Triangle A is enlarged by scale factor 3 about C at (1, 1). Where does the corner (3, 2) go?



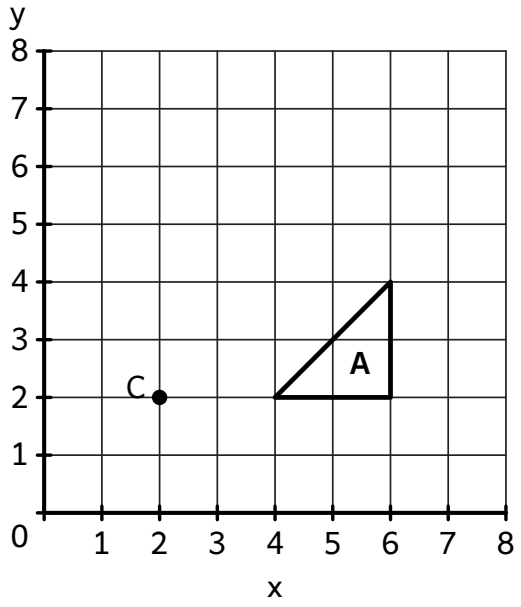
A (6, 5)

B (7, 4)

C (7, 2)

Answer at the back of the book.

2. The corner (6, 4) is enlarged by scale factor $^1/_2$ about C at (2, 2). Where does it go?



A (3, 2)

B (10, 6)

C (4, 3)

Answer at the back of the book.

3. A triangle has a side of 4 cm. It is enlarged by scale factor 3. Which statement is true?

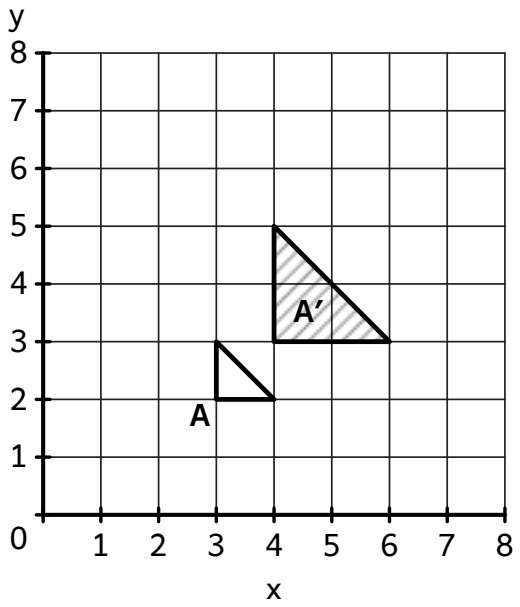
A The side is 7 cm and the angles stay the same.

B The side is 12 cm and the angles are 3 times as big.

C The side is 12 cm and the angles stay the same.

Answer at the back of the book.

4. Triangle A is enlarged to triangle A′. Where is the centre of the enlargement?



A (3, 2)

B (4, 3)

C (1, 0)

D (2, 1)

Answer at the back of the book.

5. A side of 10 cm is enlarged by scale factor $^1/_2$. How long is the image side?

A 5 cm

B 20 cm

C 9.5 cm

Answer at the back of the book.

6. **From an earlier chapter**

The point (1, 3) is translated by the vector with 5 on top and 2 below. What is the y number of the image?

Answer at the back of the book.

7. The corner (2, 5) is enlarged by scale factor 3 about O at (0, 0). What is the y number of the image?

Answer at the back of the book.

8. The corner (5, 3) is enlarged by scale factor 2 about C at (1, 1). What is the x number of the image?

   Answer at the back of the book.

9. A side of 5 cm becomes 15 cm in an enlargement. What is the scale factor?

   Answer at the back of the book.

KEEP IT

Copy this into your exercise book.

Enlarge by multiplying each distance from the centre by the scale factor. Image = centre + factor × (corner − centre). Lines through matching corners meet at the centre.

Draw: A small triangle, a bigger one, and dashed lines from the centre through matching corners.

M4.4.4 CHAPTER 4

Congruent triangles

> **Two triangles have two equal sides and one equal angle. Must they be exactly the same?**

YOUR GUESS

Pick one before you read on. It is fine to be wrong.

- ☐ **A** Yes. Three equal parts always make the triangles the same.
- ☐ **B** No, never. Two triangles with equal parts can still differ.
- ☐ **C** Not always. When the angle is between the two sides they must be. When it is not, they may differ.

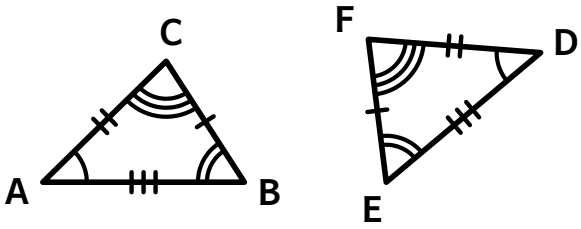
IF THIS FEELS HARD, READ THIS FIRST

- Level 3, Angles, triangles and symmetry, M3.5.5. If drawing a triangle from its sides and angles looks new, read this first.
- Level 3, Angles, triangles and symmetry, M3.5.3. If the angles of a triangle and isosceles triangles look new, read this first.

BY THE END YOU CAN

- Show two triangles are congruent using SSS, SAS, ASA or RHS
- Use congruent triangles to prove a simple fact

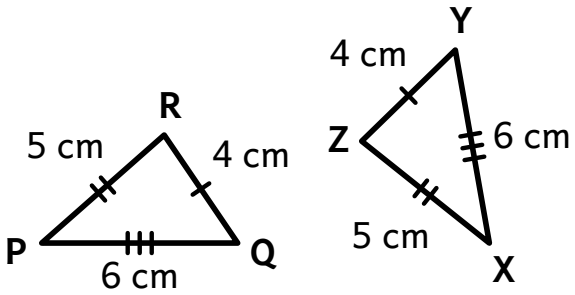
Two shapes are **congruent** when one fits exactly on the other. You may need to slide, turn or flip it first. Congruent shapes have the same size and the same form. Their **corresponding** sides and angles are equal. Those are the parts that match each other.



Matching ticks mean equal sides. Matching arcs mean equal angles.

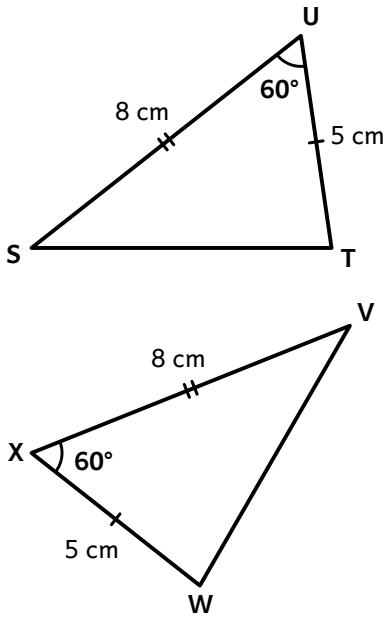
To show two triangles are congruent you do not need all six facts. Three facts, in the right pattern, are enough. There are four patterns. Each has a short name.

SSS means three sides. If all three sides of one triangle equal the three sides of another, the triangles are congruent. A triangle with sides of 4 cm, 5 cm and 6 cm can only be drawn one way.



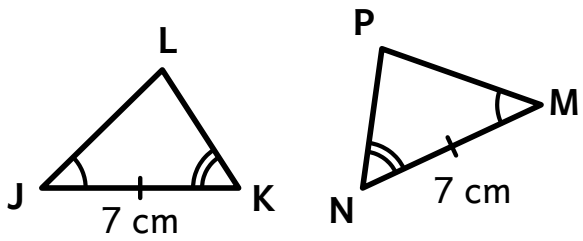
SSS: three pairs of equal sides.

SAS means two sides and the angle between them. The angle must sit between the two sides. Then the triangle is fixed, because the two sides and the angle between them decide where the third side goes.



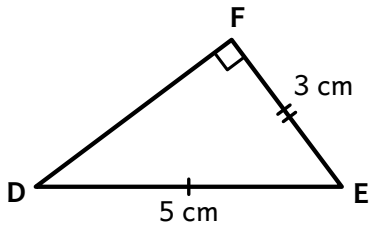
SAS: two pairs of equal sides, and the equal angles are between them.

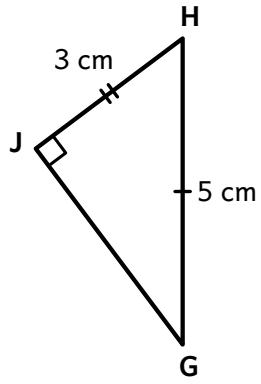
ASA means two angles and the side between them. If two angles are equal, the third angle is equal too, because the three angles of a triangle add up to 180°. So two pairs of equal angles and one pair of equal matching sides is enough, wherever that side is. This version is called AAS.



ASA: two equal angles and the side between them.

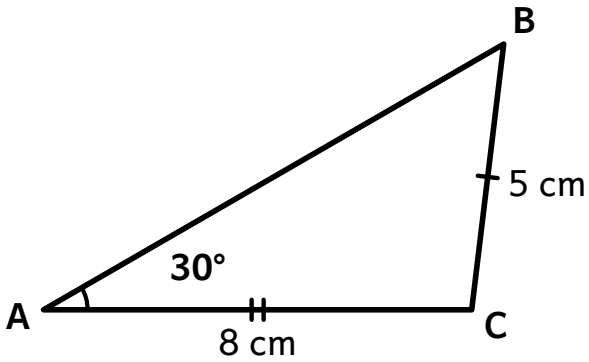
RHS is for right angled triangles only. It means a right angle, the hypotenuse and one other side. The hypotenuse is the longest side, opposite the right angle. A right angle, the hypotenuse and one more side fix the triangle.

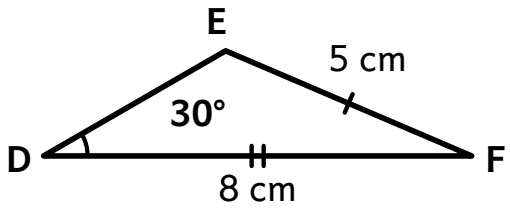




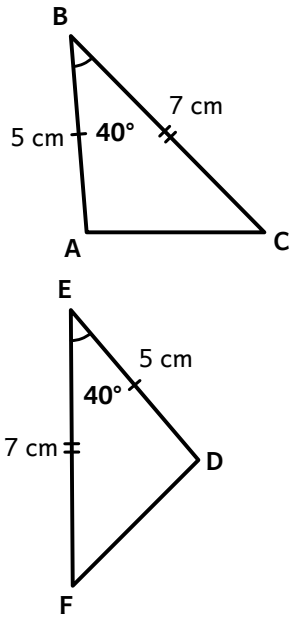
RHS: a right angle, the hypotenuse and one other side.

One pattern does not always work. Two sides and an angle that is not between them is called SSA. Look at what can happen.





SSA: the same two sides and the same angle, but two different triangles.



Triangle ABC and triangle DEF.

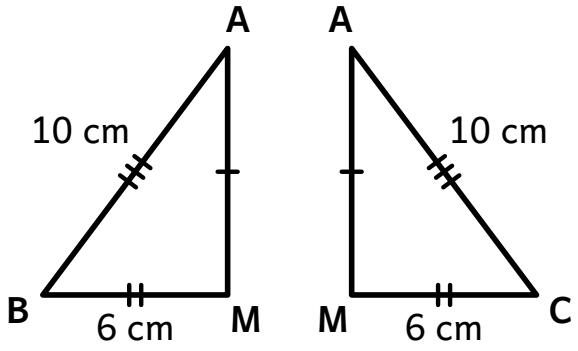
WORKED EXAMPLE

Triangle ABC has AB = 5 cm, BC = 7 cm and angle B = 40°. Triangle DEF has DE = 5 cm, EF = 7 cm and angle E = 40°. Are the triangles congruent?

1. Match the sides. AB matches DE and BC matches EF. Both pairs are equal.
2. Angle B is between AB and BC. Angle E is between DE and EF. Both angles are 40°.
3. Two sides and the angle between them are equal. So the triangles are congruent by SAS.

Answer: Yes, by SAS

Congruent triangles are useful because they prove things. A **proof** is a chain of reasons that shows a fact must be true. Take an isosceles triangle, which has two equal sides. Draw a line from the top corner to the middle of the base. It cuts the triangle into two triangles.



The two halves have three pairs of equal sides.

WORKED EXAMPLE

In triangle ABC, AB = AC = 10 cm and BC = 12 cm. M is the middle of BC. Show that triangles ABM and ACM are congruent, and find BM.

1. AB = AC = 10 cm. The problem says so.
2. BM = MC, because M is the middle of BC. Each is half of 12 cm, which is 6 cm.
3. AM is a side of both triangles, so it is equal to itself.
4. Three pairs of sides are equal, so the triangles are congruent by SSS. Also BM is 6 cm.

Answer: SSS, BM is 6 cm

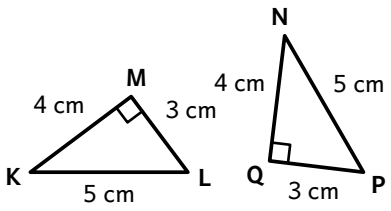
Because the triangles are congruent, their matching angles are equal. So angle ABM equals angle ACM. This proves that the angles at the base of any isosceles triangle are equal.

SEE IT THREE WAYS

With things

Cut a triangle out of paper. Trace round it on a second sheet and cut out the trace. Lay one on top of the other. They match exactly.

As a picture



In symbols

Congruent triangles have equal matching sides and equal matching angles. For example KL = NP, LM = PQ and MK = QN.

TRY IT

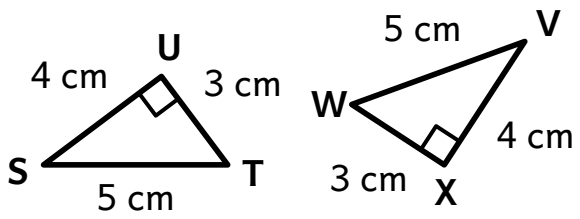
Build a triangle from three strips

You need: three strips of paper or card, 3 cm, 4 cm and 5 cm long, a friend, a pencil.

1. Lay the 3 cm and 4 cm strips so that their ends touch at a corner.
2. Swing the 5 cm strip across to join the other two ends.
3. Ask your friend to build a triangle from three strips of the same lengths. Lay one triangle on the other.

What you will see: The two triangles match exactly, however you place the strips. Three sides fix the triangle. That is SSS.

If you cannot try it, this is what happens.



A WRONG IDEA MANY PEOPLE HAVE

~~Two sides and an angle are always enough to show triangles are congruent.~~

Why it feels right: Two sides and an angle are three facts, and SAS uses exactly that.

The test that breaks it: Look at the picture of SSA. Both triangles have a side of 5 cm, a side of 8 cm and a 30° angle. But the third sides are different, so the triangles are not the same.

What is true: The angle must be between the two sides for SAS. If it is not between them, two different triangles can sometimes fit the facts, so SSA proves nothing. RHS works because the angle is 90° and the side opposite it is the longest.

KEY WORDS

congruent
having exactly the same size and form. One shape fits exactly on the other.

corresponding
matching. Corresponding sides and angles sit in the same place in each shape.

SSS
three sides of one triangle equal three sides of another.

SAS
two sides and the angle between them are equal.

ASA
two angles and the side between them are equal. If the side is not between them, the test is called AAS and it works too.

RHS
right angle, hypotenuse and one other side are equal.

proof
a chain of reasons showing a fact must be true.

BACK TO YOUR GUESS

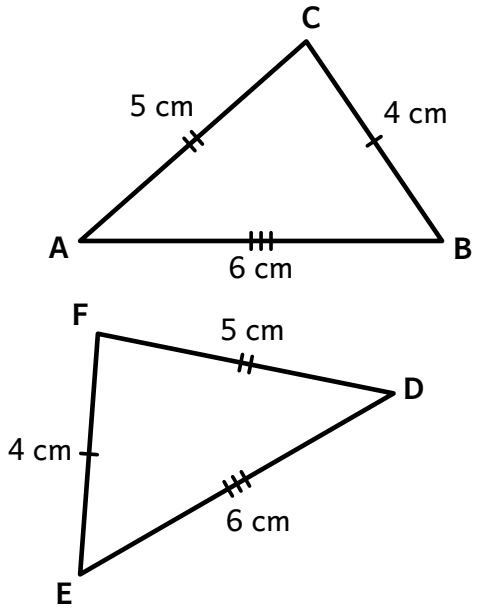
The answer is C: Not always. When the angle is between the two sides they must be. When it is not, they may differ.

It depends where the angle is. When the equal angle is between the two equal sides, the triangles are the same. When it is not, two different triangles can sometimes be made, so this pattern, SSA, is not a test. RHS is the one exception you will use: a right angle with the hypotenuse fixes the triangle. This chapter shows which sets of three facts are enough.

CHECK YOURSELF

Try each one before you look. A wrong answer tells you where to go next.

1. The marks show that three pairs of sides are equal. Which test shows that the triangles are congruent?



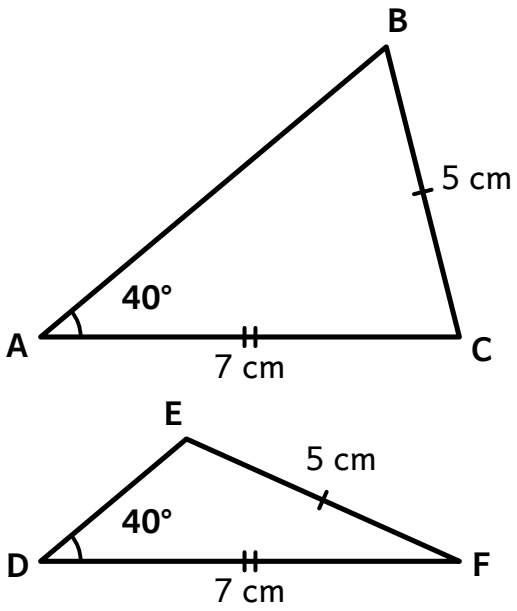
A SAS

B SSS

C RHS

Answer at the back of the book.

2. These two triangles have two equal sides and an equal angle. Are they congruent?



A Yes, by SAS.

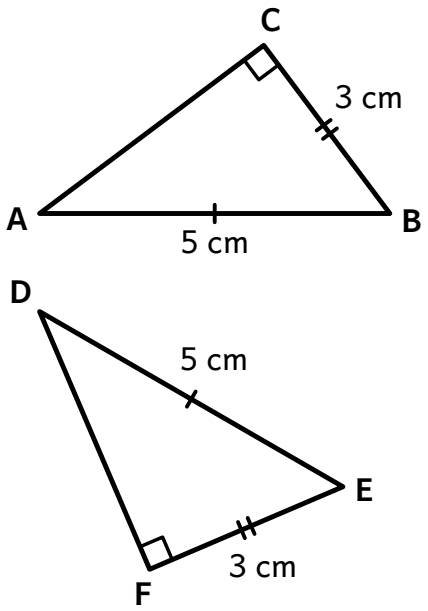
B Yes, by SSS.

C Yes, because they look alike.

D No. Their third sides are different lengths. SSA cannot prove two triangles congruent.

Answer at the back of the book.

3. Each triangle has a right angle. The marks show equal hypotenuses and one other equal side. Which test shows that the triangles are congruent?



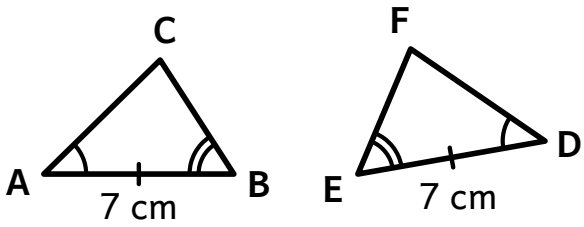
A ASA

B SAS

C RHS

Answer at the back of the book.

4. The marks show two equal angles and the side between them. Which test shows that the triangles are congruent?



A ASA

B SSS

C RHS

Answer at the back of the book.

5. Triangles ABM and ACM are congruent. AB = AC = 13 cm and BC = 10 cm. M is the middle of BC. How long is BM?

   A 13 cm

   B 10 cm

   C 5 cm

   Answer at the back of the book.

6. **From an earlier chapter**

   A side of 6 cm is enlarged by scale factor 4. How long is the image side, in centimetres?

   Answer at the back of the book.

7. Triangle ABC has sides of 5 cm, 6 cm and 9 cm. Triangle DEF is congruent to it and has sides of 6 cm, 9 cm and x cm. What is x?

   Answer at the back of the book.

8. Triangle ABC has AB = AC and angle B = 70°. Triangles ABM and ACM are congruent, so angle C is also 70°. What is angle A, in degrees?

   Answer at the back of the book.

9. Triangle ABC has angles of 50° and 60°. Triangle DEF also has angles of 50° and 60°, and one pair of matching sides is equal. What is the third angle of each triangle, in degrees?

   Answer at the back of the book.

KEEP IT

Copy this into your exercise book.

Congruent triangles match exactly. Tests: SSS, SAS (angle between), ASA, RHS. SSA is not a test, but RHS works. Use them to prove things, such as equal base angles.

Draw: Two matching triangles with ticks on the equal sides and arcs on the equal angles.

Similar shapes

> **A 3 m pole has a 2 m shadow. At the same time a tree has a 6 m shadow. How tall is the tree?**

YOUR GUESS

Pick one before you read on. It is fine to be wrong.

- ☐ **A** 6 m, because the tree is as tall as its shadow.
- ☐ **B** 7 m, because the shadow is 4 m longer, so add 4 to 3.
- ☐ **C** 9 m, because the shadow is 3 times as long.

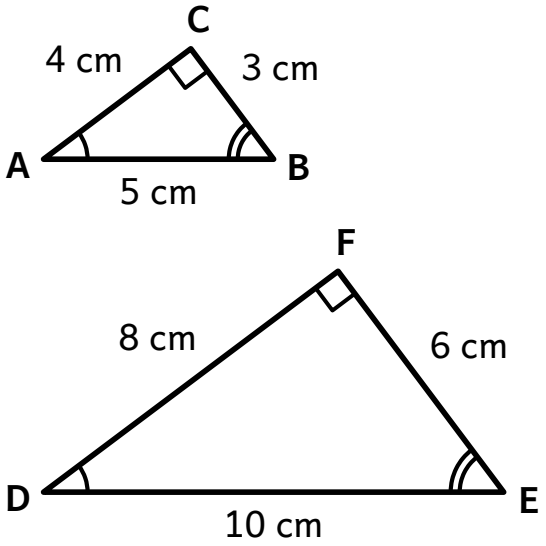
IF THIS FEELS HARD, READ THIS FIRST

- Level 3, Ratio and proportion, M3.4.4. If working out one and then many looks new, read this first.
- Level 3, Angles, triangles and symmetry, M3.5.3. If the angles of a triangle adding to 180° look new, read this first.

BY THE END YOU CAN

- Show two triangles are similar from their angles
- Find a missing side using a scale factor

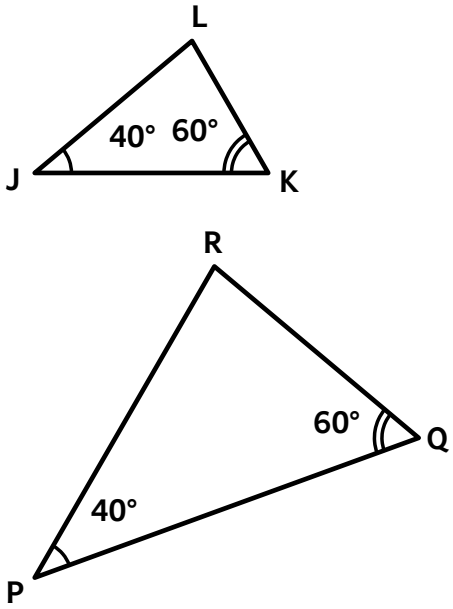
Two shapes are **similar** when one is an enlargement of the other. The enlargement may also turn or flip the shape. Similar shapes have the same form but not always the same size. Each pair of matching angles is equal. Each pair of matching sides is in the same ratio.



The sides of the large triangle are 2 times the matching sides of the small one.

Take the triangles above. Divide each large side by its matching small side: 6 ÷ 3 = 2, 8 ÷ 4 = 2 and 10 ÷ 5 = 2. All three answers are 2. That number is the scale factor from the small triangle to the large one.

To show two triangles are similar, you only need their angles. The three angles of a triangle add up to 180°. If two angles match, the third is found by taking them from 180°, so it matches too.

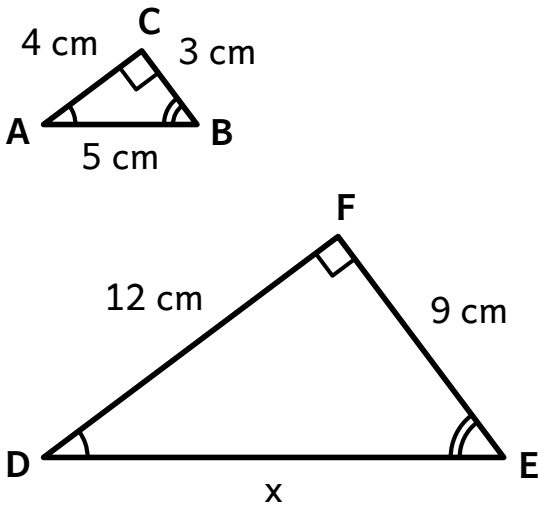


Two angles match, so the third angle is 80° in both. The triangles are similar.

This test works for triangles. For shapes with more sides, equal angles are not enough. A rectangle that is 2 cm by 4 cm and one that is 2 cm by 8 cm have only

right angles. Their sides are not in the same ratio, so they are not similar.

Now use the scale factor to find a missing side. Pair up the matching sides. Divide a large side by its small match to get the scale factor. Then multiply any small side by it.



Triangle DEF is similar to triangle ABC. Side x matches the 5 cm side.

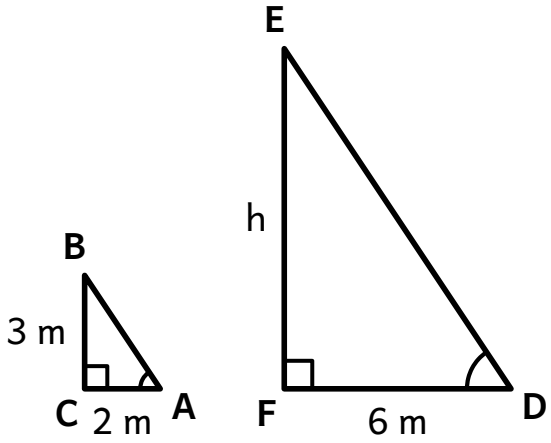
WORKED EXAMPLE

Triangle ABC has sides of 3 cm, 4 cm and 5 cm. Triangle DEF is similar. Its side of 9 cm matches the 3 cm side. How long is the side that matches 5 cm?

1. Find the scale factor. Divide the large side by its small match: 9 ÷ 3 = 3.
2. Multiply the 5 cm side by the scale factor: 5 × 3 = 15.
3. The side that matches 5 cm is 15 cm long.

Answer: 15 cm

Similar triangles solve real problems. On level ground, an upright pole and an upright tree both make a right angle with the ground. At the same time the sun is at the same angle for both. So two angles match, and the shadow triangles are similar.



The pole triangle ABC is on the left and the tree triangle DEF is on the right. The sun makes the same angle with the ground for both.

WORKED EXAMPLE

A pole 3 m high has a shadow 2 m long. At the same time a tree has a shadow 6 m long. How high is the tree?

1. The pole and the tree are both upright, and the sun is at the same angle. The shadow triangles have equal angles, so they are similar.
2. The scale factor from the pole shadow to the tree shadow is 6 ÷ 2 = 3.
3. The tree is 3 times as high as the pole: 3 × 3 = 9 m.

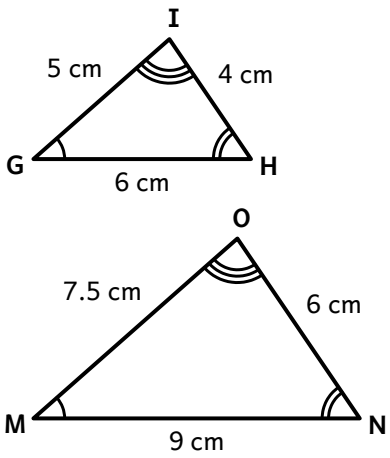
Answer: 9 m

SEE IT THREE WAYS

With things

Print a photo small and then large. Every person in it looks the same, only bigger. Each length in the large print is the same number of times its length in the small one.

As a picture



In symbols

Scale factor = 6 ÷ 4 = 1.5. Then 5 × 1.5 = 7.5 and 6 × 1.5 = 9. Every side is multiplied by 1.5.

TRY IT

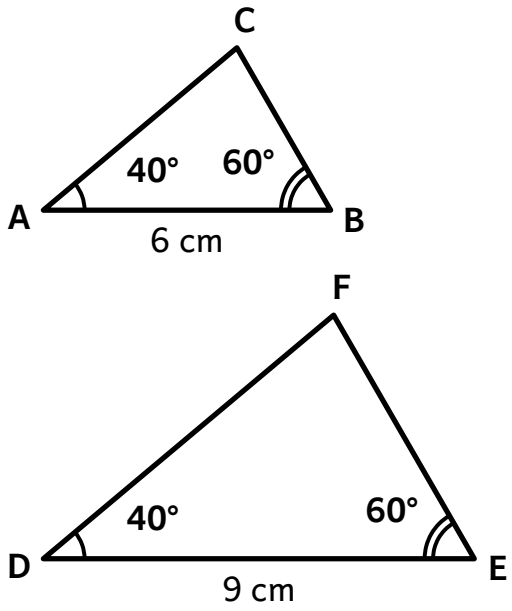
Draw two triangles with the same angles

You need: a ruler, a protractor, a pencil, plain paper.

1. Draw a line 6 cm long. At one end draw an angle of 40°. At the other end draw an angle of 60°. Extend the two lines until they meet.
2. Draw a second line, 9 cm long. Use the same two angles, 40° and 60°.
3. Measure the sides that lie opposite the 40° angle in each triangle. Divide the larger by the smaller.
4. Do the same for the sides opposite the 60° angle.

What you will see: Both answers are 1.5, the same as 9 ÷ 6. Triangles with the same angles have sides in the same ratio.

If you cannot try it, this is what happens.



A WRONG IDEA MANY PEOPLE HAVE

~~If one side grows from 3 cm to 9 cm, then every side grows by 6 cm.~~

Why it feels right: Adding is how we often compare two numbers. The number 9 is 6 more than 3.

The test that breaks it: Add 6 to the 5 cm side and it becomes 11 cm. The sides would be 9 cm, 10 cm and 11 cm. These are not in the same ratio as 3 cm, 4 cm and 5 cm, so the shape changes.

What is true: Similar shapes grow by multiplying. The scale factor is 9 ÷ 3 = 3, so the 5 cm side becomes 5 × 3 = 15 cm.

KEY WORDS

similar
having the same form, with equal matching angles and matching sides in the same ratio. One shape is an enlargement of the other.

BACK TO YOUR GUESS

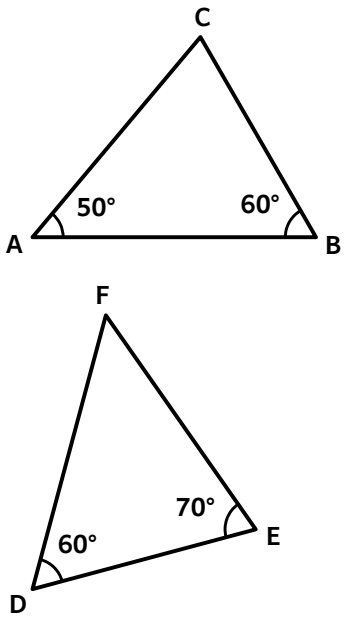
The answer is C: 9 m, because the shadow is 3 times as long.

The pole and the tree both stand upright on level ground, and the sun is at the same angle for both. So their shadow triangles have the same shape. The tree shadow is 3 times the pole shadow, so the tree is 3 times the pole: 9 m.

CHECK YOURSELF

Try each one before you look. A wrong answer tells you where to go next.

1. Two angles are marked in each triangle. Are the triangles similar?



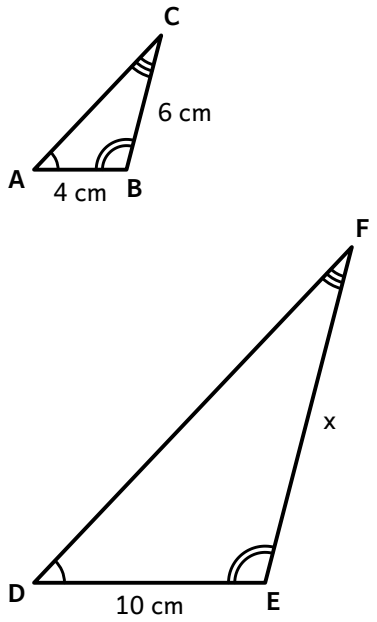
A Yes. The missing angles are 70° and 50°, so all three angles match.

B No. The marked angles are different.

C Yes, because both shapes are triangles.

Answer at the back of the book.

2. Triangles ABC and DEF are similar. AB = 4 cm matches DE = 10 cm. BC = 6 cm. How long is EF?



A 12 cm

B 2.4 cm

C 15 cm

Answer at the back of the book.

3. Triangle PQR is a smaller copy of triangle ABC. AB = 12 cm matches PQ = 8 cm. BC = 18 cm. How long is QR?

A 27 cm

B 12 cm

C 14 cm

Answer at the back of the book.

4. Which pair of shapes must be similar?

A Two rectangles

B Two triangles that each have one angle of 40°

C Two squares

D Two kites

Answer at the back of the book.

5. A 2 m stick has a shadow 3 m long. At the same time a building has a shadow 24 m long. How tall is the building?

   A 36 m

   B 16 m

   C 23 m

   Answer at the back of the book.

6. **From an earlier chapter**

   Triangle ABC has AB = AC and angle B = 65°. What is angle A, in degrees?

   Answer at the back of the book.

7. Triangle ABC has sides of 5 cm and 7 cm. A similar triangle has a side of 15 cm that matches the 5 cm side. How long is the side that matches 7 cm, in centimetres?

   Answer at the back of the book.

8. A triangle has sides of 12 cm and 15 cm. A similar, smaller triangle has a side of 4 cm that matches the 12 cm side. How long is the side that matches 15 cm, in centimetres?

   Answer at the back of the book.

9. A photo is 12 cm wide and 8 cm high. It is enlarged so that it is 30 cm wide. How high is the enlarged photo, in centimetres?

   Answer at the back of the book.

KEEP IT

Copy this into your exercise book.

Similar shapes have equal angles and matching sides in the same ratio. Two equal angles make triangles similar. Scale factor = a side of one ÷ the matching side of the other. Multiply.

Draw: A small and a large triangle with the same angle arcs, and the scale factor written between them.

M4.4.6 CHAPTER 6

Similar areas and volumes

> **A jar is made twice as high and twice as wide. Does it hold twice as much?**

YOUR GUESS

Pick one before you read on. It is fine to be wrong.

- ☐ **A** Yes. Twice the size holds twice the amount.
- ☐ **B** It holds 4 times as much.
- ☐ **C** It holds 8 times as much.

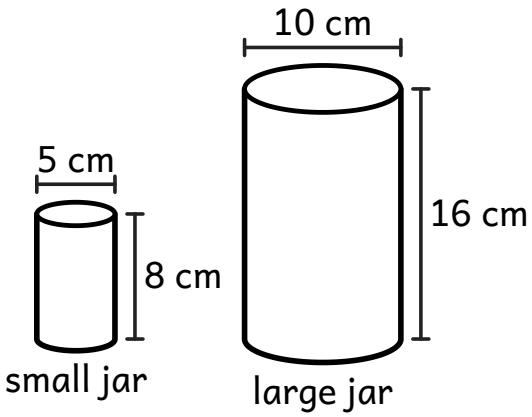
IF THIS FEELS HARD, READ THIS FIRST

- Level 3, Perimeter, area and volume, M3.6.3. If finding an area by multiplying looks new, read this first.
- Level 3, Perimeter, area and volume, M3.6.7. If volume as length × width × height looks new, read this first.

BY THE END YOU CAN

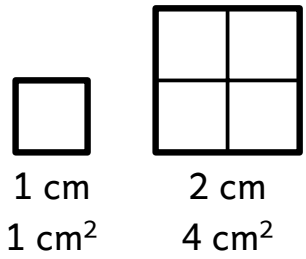
- Use k, k^2 and k^3 for length, area and volume scale factors
- Find the volume of a larger jar of the same shape

Two solids of the same shape are similar, like two jars made from the same mould. Every length of the large jar is the same number of times the matching length of the small jar. That number is the scale factor, k. Here k = 2.



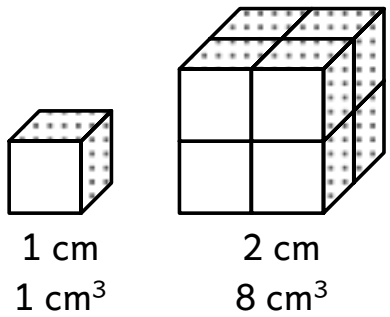
Every length of the large jar is 2 times the matching length of the small jar.

Lengths go up by k. Areas and volumes do not go up by k. Start with areas.



The sides are 2 times as long. The area is 4 times as big.

The big square has 2 rows of 2 small squares, so 2 × 2 = 4 of them. For scale factor k, the big square has k rows of k small squares, so k × k = k^2 of them. Any flat shape can be cut into small squares, so every area is multiplied by k^2. This is the ***area scale factor***.

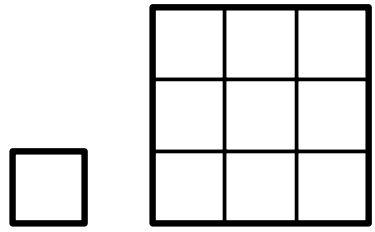


The edges are 2 times as long. The volume is 8 times as big.

The big block has 2 layers. Each layer has 2 rows of 2 small cubes. That is 2 × 2 × 2 = 8 cubes. For scale factor k there are k × k × k = k^3 small cubes. Every volume is multiplied by k^3. This is the **volume scale factor**.

k	Len.	Area	Vol.
2	× 2	× 4	× 8
3	× 3	× 9	× 27
10	× 10	× 100	× 1 000

Lengths multiply by k, areas by k × k, volumes by k × k × k.



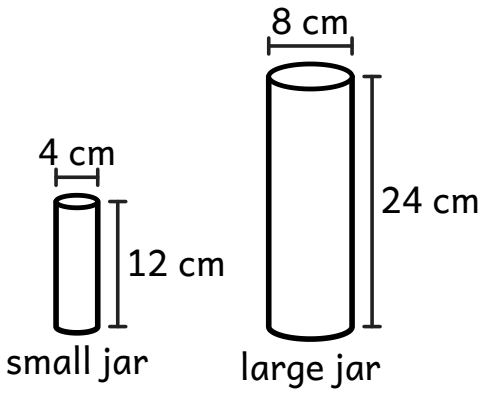
With k = 3, there are 9 small squares in the big square.

WORKED EXAMPLE

A photo frame has an area of 20 cm². A similar frame has all its lengths 3 times as long. What is the area of the larger frame?

1. The length scale factor is k = 3.
2. The area scale factor is $k^2 = 3 \times 3 = 9$.
3. The new area is 20 × 9 = 180 cm².

Answer: 180 cm²



The large jar is twice as high and twice as wide.

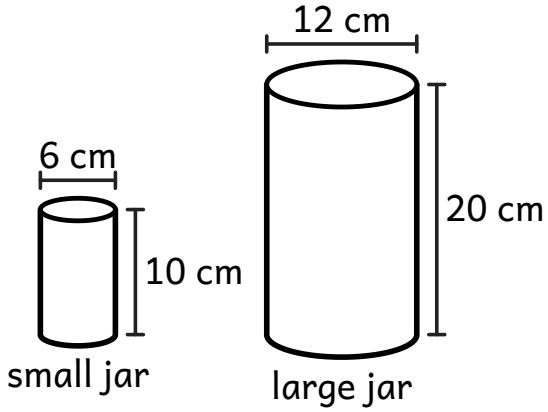
WORKED EXAMPLE

A small jar holds about 150 ml. A larger jar of the same shape is twice as high. About how much does the larger jar hold?

1. Twice as high means the length scale factor is k = 2.
2. The volume scale factor is $k^3 = 2 \times 2 \times 2 = 8$.
3. The larger jar holds about $150 \times 8 = 1\,200$ ml.

Answer: about 1 200 ml

You can also go backwards. If you know how areas compare, take a square root to find k. If you know how volumes compare, take a cube root.



The whole outsides of the jars have areas of about 245 cm^2 and 980 cm^2.

WORKED EXAMPLE

Two jars have the same shape. Their whole outsides, with lid and base, have areas of about 245 cm² and 980 cm². The small jar holds about 283 ml. Find k and the volume of the large jar.

1. The area scale factor is 980 ÷ 245 = 4.
2. The area scale factor is k^2, so k is the square root of 4, which is 2.
3. The volume scale factor is $k^3 = 2 \times 2 \times 2 = 8$.
4. The large jar holds about 283 × 8 = 2 264 ml.

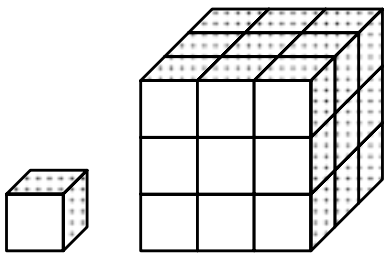
Answer: k is 2 and the large jar holds about 2 264 ml

SEE IT THREE WAYS

With things

Build a cube from 27 small cubes, 3 along each edge. Its edge is 3 times the edge of one small cube. Count the small cubes: 3 × 3 × 3.

As a picture



1 cm
1 cm^3

3 cm
27 cm^3

In symbols

Scale factor k for lengths. Scale factor k × k for areas. Scale factor k × k × k for volumes. With k = 3 these are 3, 9 and 27.

TRY IT

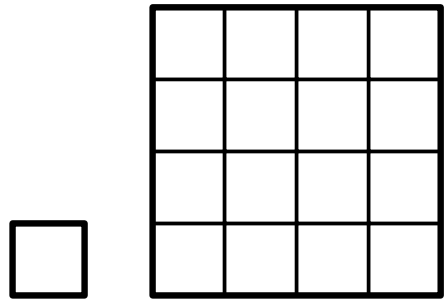
Count the squares

You need: squared paper, a pencil.

1. Draw a square 1 cm by 1 cm. Draw a square 4 cm by 4 cm beside it.
2. Count the 1 cm squares inside the big square.
3. Divide the number of squares in the big square by the number in the small square.

What you will see: The big square has 16 small squares. The sides are 4 times as long, and 4 × 4 = 16.

If you cannot try it, this is what happens.



A WRONG IDEA MANY PEOPLE HAVE

~~If a shape is enlarged by scale factor 3, its area is 3 times as big.~~

Why it feels right: The lengths are 3 times as long, so it seems the area should be 3 times as big too.

The test that breaks it: Take a square 1 cm by 1 cm with an area of 1 cm^2. Enlarge it by scale factor 3. It becomes 3 cm by 3 cm, with an area of 9 cm^2. That is 9 times as big, not 3 times.

What is true: Area is length times width, and both lengths are multiplied by k. So the area is multiplied by $k \times k = k^2$. Volume has three lengths, so it is multiplied by k^3.

KEY WORDS

area scale factor
the number every area is multiplied by. It is k^2, where k is the length scale factor.

volume scale factor
the number every volume is multiplied by. It is k^3, where k is the length scale factor.

BACK TO YOUR GUESS

The answer is C: It holds 8 times as much.

Doubling the height, the width and the depth makes 2 × 2 × 2 = 8 times the space inside. A jar that is twice as big in every direction holds 8 times as much. This chapter shows why.

CHECK YOURSELF

Try each one before you look. A wrong answer tells you where to go next.

1. A small box has a volume of 40 cm^3. A similar large box has every length 3 times as long. What is the volume of the large box?

   A 120 cm^3

   B 360 cm^3

   C 1 080 cm^3

   Answer at the back of the book.

2. A poster has an area of 50 cm^2. It is enlarged by scale factor 4. What is the area of the enlarged poster?

   A 200 cm^2

   B 800 cm^2

   C 3 200 cm^2

   Answer at the back of the book.

3. Two similar shapes have areas of 12 cm^2 and 108 cm^2. What is the length scale factor from the small shape to the large one?

   A 3

   B 9

   C 96

   Answer at the back of the book.

4. A small jar holds 100 ml. A similar large jar holds 800 ml. How many times as high as the small jar is the large jar?

   A 8

   B 4

   C 2

   Answer at the back of the book.

5. A shape is enlarged by scale factor 5. Which statement about its area is true?

   A The area is 5 times as big.

   B The area is 25 times as big.

   C The area is 10 times as big.

   Answer at the back of the book.

6. **From an earlier chapter**

   Triangle ABC has sides of 4 cm and 5 cm. A similar triangle has a side of 12 cm that matches the 4 cm side. How long is the side that matches 5 cm, in centimetres?

   Answer at the back of the book.

7. A shape has an area of 6 cm^2. It is enlarged by scale factor 3. What is the new area, in cm^2?

   Answer at the back of the book.

8. A jar holds 15 ml. A jar of the same shape has every length 2 times as long. How many ml does it hold?

   Answer at the back of the book.

9. A small jar holds 10 ml. A similar large jar holds 270 ml. How many times as high as the small jar is the large jar?

   Answer at the back of the book.

KEEP IT

Copy this into your exercise book.

Scale factor k: lengths × k, areas × k^2, volumes × k^3. Doubling every length gives 4 times the area and 8 times the volume.

Draw: A 1 cm cube beside a 2 cm cube made of 8 small cubes, with the three scale factors written under them.

M4.4.7 CHAPTER 7

Arcs and sectors

> **A quarter of a round flat bread has a radius of 10 cm. How long is its curved edge?**

YOUR GUESS

Pick one before you read on. It is fine to be wrong.

- ☐ **A** About 62.8 cm. That is the whole distance round the circle.
- ☐ **B** About 15.7 cm. That is a quarter of the distance round the circle.
- ☐ **C** About 25 cm. A quarter of 100 cm.

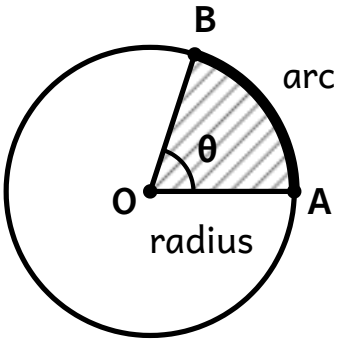
IF THIS FEELS HARD, READ THIS FIRST

- Level 3, Perimeter, area and volume, M3.6.6. If the circumference $2\pi r$ and the area πr^2 look new, read this first.
- Level 3, Angles, triangles and symmetry, M3.5.1. If a full turn of 360° looks new, read this first.

BY THE END YOU CAN

- Find the length of an arc of a circle
- Find the area of a sector
- Find the perimeter of a sector

A circle can be cut like a flat bread. A piece of the curved edge is an ***arc***. A slice, with two straight edges and an arc, is a **sector**. The straight edges of a sector are both radii. They meet at the centre, O, and make an angle there.

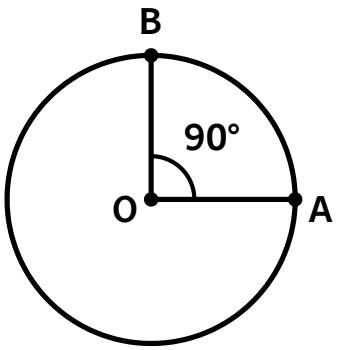


The shaded part is a sector. The heavy curve is its arc. The angle θ is at the centre.

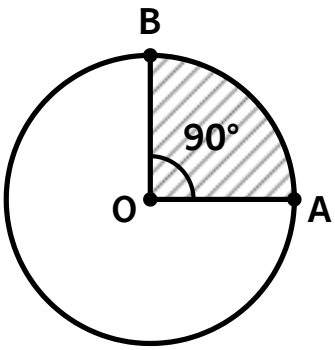
A full turn is 360°. A sector with an angle of θ degrees is θ out of 360 of the whole circle. If θ = 90°, the sector is 90 out of 360, which is a quarter. If θ = 180°, it is a half.

DRAW IT FIRST

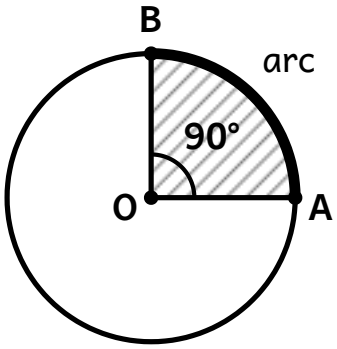
A sector of 90° is a quarter of the circle



1 Draw two radii, OA and OB, with an angle of 90° between them.



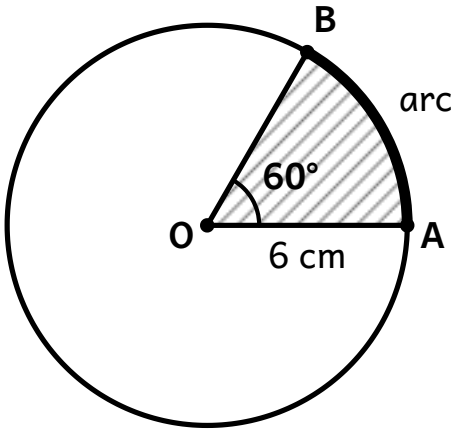
2 Shade the sector between the radii. It is 90 out of 360 of the circle, which is a quarter.



3 The arc is the part of the circumference inside the sector. It is a quarter of the circumference.

This gives the rules. The circumference is $2\pi r$ and the area is πr^2. A sector takes the fraction $\theta \div 360$ of each. Use the π key on your calculator, and say how you rounded.

Arc length = $(\theta \div 360) \times 2\pi r$. Sector area = $(\theta \div 360) \times \pi r^2$. Perimeter of a sector = arc length + 2r. The 2r is for the two straight edges.



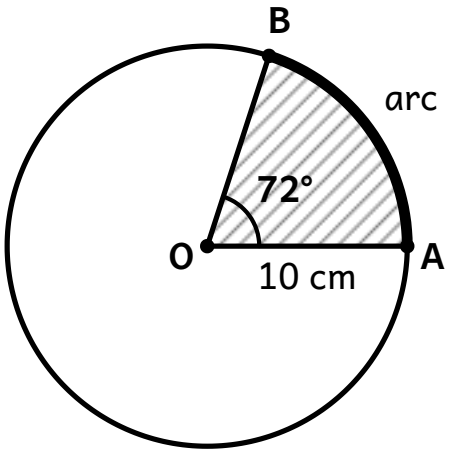
A sector with a radius of 6 cm and an angle of 60°.

WORKED EXAMPLE

Find the length of the arc of a sector with a radius of 6 cm and an angle of 60°. Give the answer to 1 decimal place.

1. To 1 decimal place, the whole circumference is $2 \times \pi \times 6 = 37.7$ cm.
2. The angle 60° is 60 out of 360. So the arc is 60 ÷ 360 of the circumference.
3. To 1 decimal place, the arc length is $60 \div 360 \times 37.7 = 6.3$ cm.

Answer: 6.3 cm



A sector with a radius of 10 cm and an angle of 72°.

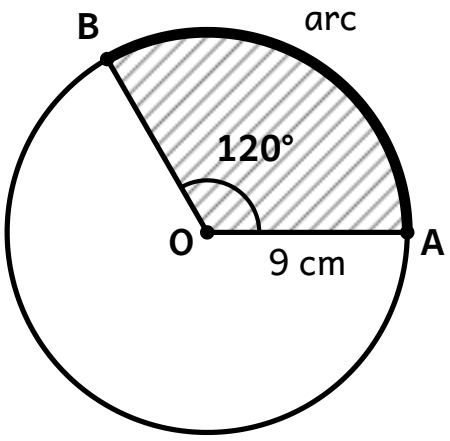
WORKED EXAMPLE

Find the area of a sector with a radius of 10 cm and an angle of 72°. Give the answer to 1 decimal place.

1. To 1 decimal place, the area of the whole circle is $\pi \times 10 \times 10 = 314.2$ cm^2.
2. The sector is 72 out of 360 of the circle. That is $72 \div 360 = 0.2$.
3. To 1 decimal place, the sector area is $0.2 \times 314.2 = 62.8$ cm^2.

Answer: 62.8 cm^2

To walk all the way round the edge of a sector, you go along the arc and then along both radii. So the perimeter is longer than the arc.



A sector with a radius of 9 cm and an angle of 120°.

WORKED EXAMPLE

Find the perimeter of a sector with a radius of 9 cm and an angle of 120°. Give the answer to 1 decimal place.

1. To 1 decimal place, the arc length is $120 \div 360 \times 2 \times \pi \times 9 = 18.8$ cm.
2. The sector has two straight edges. Each is a radius of 9 cm.
3. Together the straight edges are $2 \times 9 = 18$ cm.
4. Perimeter = $18.8 + 18 = 36.8$ cm.

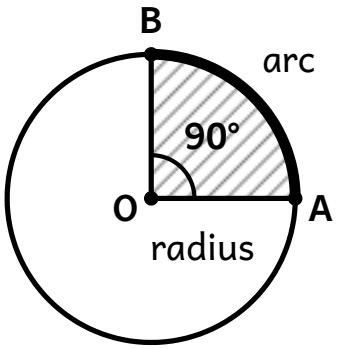
Answer: 36.8 cm

SEE IT THREE WAYS

With things

Cut a round piece of paper. Fold it in half, then in half again. Open it. Each fold line is a radius, and a quarter of the circle lies between two of them.

As a picture



In symbols

Arc length = (θ ÷ 360) × 2πr. Sector area = (θ ÷ 360) × πr². Sector perimeter = arc length + 2r.

TRY IT

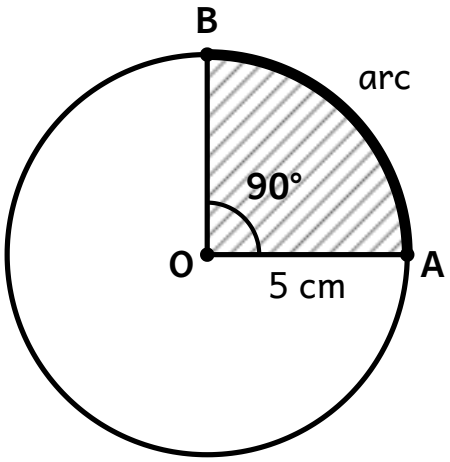
Measure an arc with string

You need: a round tin lid or a compass, string, a ruler, a pencil.

1. Draw a circle with a radius of 5 cm. Mark its centre.
2. Draw two radii at right angles to each other. Check the angle with the corner of a sheet of paper.
3. Lay a piece of string along the arc between the radii. Straighten the string and measure it.

What you will see: The string is about 7.9 cm long. That matches 90 ÷ 360 × 2 × π × 5, which is about 7.85 cm.

If you cannot try it, this is what happens.



A WRONG IDEA MANY PEOPLE HAVE

~~The perimeter of a sector is the length of its arc.~~

Why it feels right: The arc is the curved edge, and the curved edge is the part that needs the new rule.

The test that breaks it: Trace round a slice of flat bread with a finger. You go along the curve, then along one straight edge, then along the other. A sector with a radius of 9 cm has 18 cm of straight edges.

What is true: The perimeter is the whole distance round the edge. It is the arc length plus two radii: arc + 2r.

KEY WORDS

arc
a part of the circumference of a circle.

sector
a slice of a circle between two radii. It has an arc for its curved edge.

BACK TO YOUR GUESS

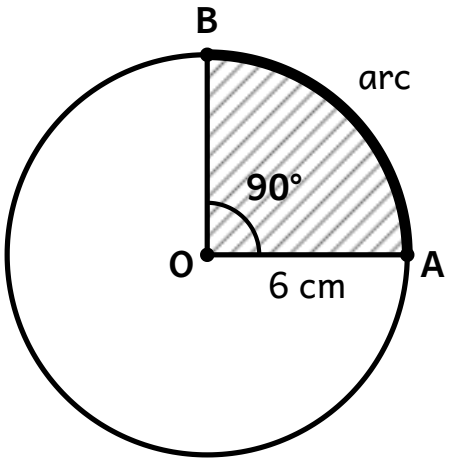
The answer is B: About 15.7 cm. That is a quarter of the distance round the circle.

The curved edge is a quarter of the circumference. The circumference is about 62.8 cm, because 2 × π × 10 is about 62.8. A quarter of that is about 15.7 cm. This chapter shows how to do the same for any slice.

CHECK YOURSELF

Try each one before you look. A wrong answer tells you where to go next.

1. A sector has a radius of 6 cm and an angle of 90°. How long is its arc, to 1 decimal place?



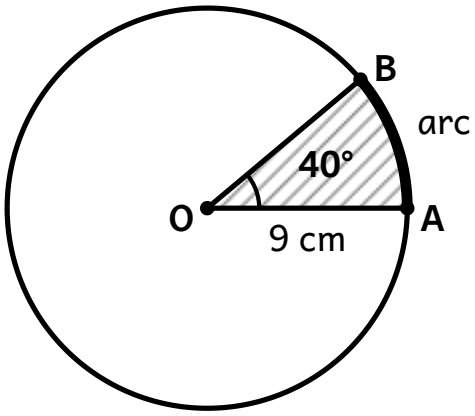
A 37.7 cm

B 9.4 cm

C 28.3 cm

Answer at the back of the book.

2. A sector has a radius of 9 cm and an angle of 40°. What is its area, to 1 decimal place?



A 254.5 cm^2

B 6.3 cm^2

C 28.3 cm^2

Answer at the back of the book.

3. A sector has a radius of 5 cm and an angle of 90°. What is its perimeter, to 1 decimal place?

A 17.9 cm

B 7.9 cm

C 12.9 cm

Answer at the back of the book.

4. Which rule gives the length of an arc?

A Add the angle to the radius.

B Multiply the angle in degrees by the radius.

C Multiply the fraction of the circle by the area of the whole circle.

D Multiply the fraction of the circle by the circumference.

Answer at the back of the book.

5. A sector has a radius of 4 m and an angle of 90°. What is its area, to 1 decimal place?

   A 50.3 m^2

   B 12.6 m^2

   C 12.6 m

   Answer at the back of the book.

6. **From an earlier chapter**

   A shape has an area of 7 cm^2. It is enlarged by scale factor 3. What is the new area, in cm^2?

   Answer at the back of the book.

7. A sector has a radius of 12 cm and an angle of 90°. How long is its arc, to 1 decimal place?

   Answer at the back of the book.

8. A sector has a radius of 6 cm and an angle of 150°. What is its area, in cm^2, to 1 decimal place?

   Answer at the back of the book.

9. A sector has a radius of 8 cm and an angle of 45°. What is its perimeter, in cm, to 1 decimal place?

   Answer at the back of the book.

KEEP IT

Copy this into your exercise book.

Arc = (θ ÷ 360) × 2πr. Sector area = (θ ÷ 360) × πr^2. Sector perimeter = arc + 2r. Say how you rounded.

Draw: A circle with a shaded sector, its angle at the centre, the radius labelled, and the arc drawn heavy.

Angle rules in circles

> **Two lines from the ends of an arc meet at the centre at 100°. What is the angle when they meet at the edge instead?**

YOUR GUESS

Pick one before you read on. It is fine to be wrong.

- ☐ **A** 100°, the same as at the centre.
- ☐ **B** 50°, half of the angle at the centre.
- ☐ **C** 200°, double the angle at the centre.

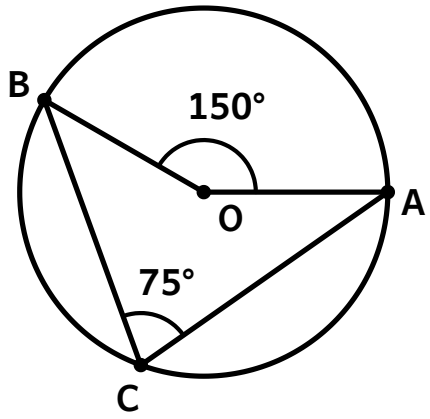
IF THIS FEELS HARD, READ THIS FIRST

- Level 3, Angles, triangles and symmetry, M3.5.3. If the angles of a triangle and isosceles triangles look new, read this first.
- Level 3, Angles, triangles and symmetry, M3.5.4. If the angles of a quadrilateral adding to 360° look new, read this first.

BY THE END YOU CAN

- Use the rule that the angle at the centre is double
- Find angles in a semicircle and in a cyclic quadrilateral
- Use the rule that a tangent meets a radius at 90°

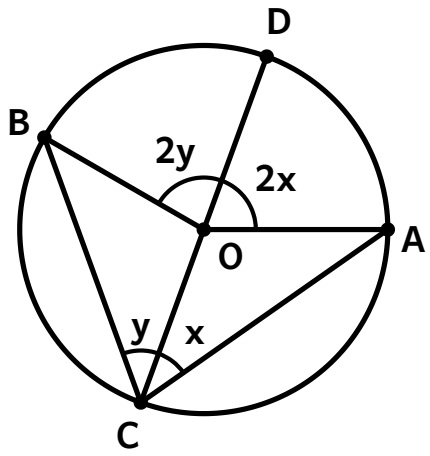
A **chord** is a straight line joining two points on a circle. The two points cut the circle into two arcs. A line from the centre to each end of a chord makes an angle at the centre. A line from a third point on the circle to each end makes an angle at the circumference. Both angles **stand on** the arc AB that does not contain C.



The angle at the centre is 150°. The angle at the circumference is 75°. They stand on the same arc AB.

The first rule. The angle at the centre is double the angle at the circumference on the same arc. Here 150° is double 75°. Measure with a protractor on a big drawing and you will see it holds for any circle.

Here is why. Draw the line from C through O to the other side of the circle, ending at D. The lines OA, OB and OC are all radii, so they are equal. That makes triangles OAC and OBC isosceles.

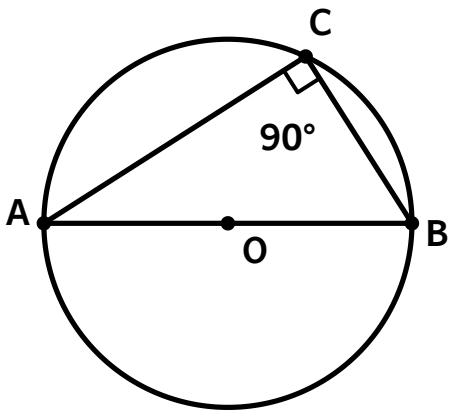


The line CD splits both angles into two parts. The angles x and y are at C. The angles 2x and 2y are at O.

Call the angle at C in triangle OAC x. Chapter 4 proved that the base angles of an isosceles triangle are equal, so the angle at A is also x. The angles on the straight line CD add to 180°, and so do the angles of triangle OAC. So the angle AOD is x + x = 2x. The same reasoning on the other side gives angle DOB = 2y. The angle at C is x + y and the angle at O is 2x + 2y. That is double. This drawing has O inside the angle

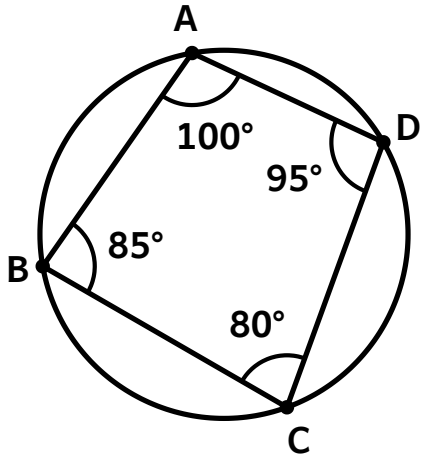
at C. When O lies outside it, the same steps work with a subtraction in place of the addition.

The second rule follows at once. A diameter makes a straight angle of 180° at the centre. The angle at the circumference on the same arc is half of that, which is 90°. So the angle in a **semicircle** is 90°.



AB is a diameter. The angle at C is 90°, wherever C is on the circle.

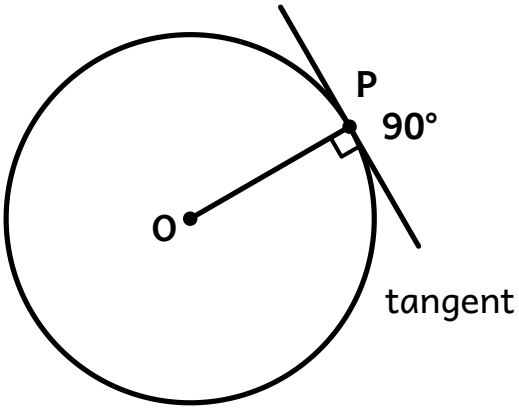
The third rule is about a **cyclic quadrilateral**. It is a four sided shape with all four corners on a circle. Opposite angles add up to 180°. Why? The two angles at the centre on the arcs between the opposite corners add to 360°. Each angle at the edge is half of its centre angle. So the two edge angles add to half of 360°, which is 180°.



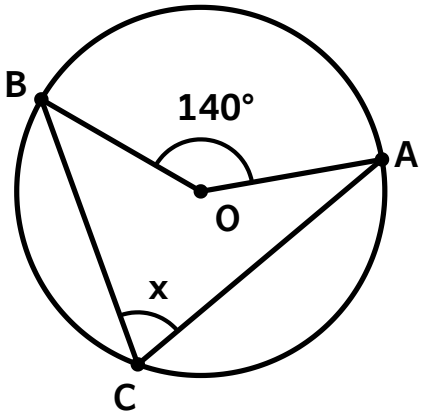
Opposite corners: 100° + 80° = 180°, and 85° + 95° = 180°.

The fourth rule is about a **tangent**. A tangent is a straight line that touches a circle at one point. Every other point of the line lies outside the circle. So the radius to the

touching point is the shortest line from the centre to the tangent. The shortest line to a line is the one at a right angle. So a tangent meets the radius at 90°.



The tangent touches the circle at P. The radius OP is at 90° to it.



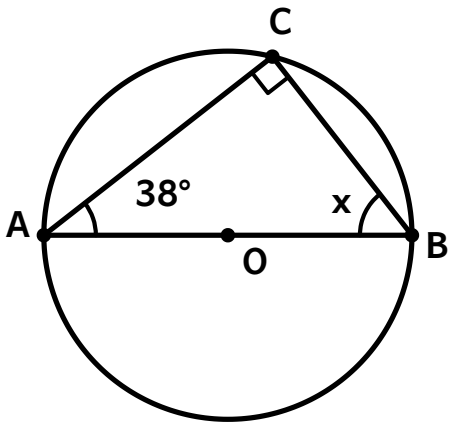
Find the angle x.

WORKED EXAMPLE

The angle at the centre of a circle is 140°. Find the angle x at the circumference, on the same arc.

1. Both angles stand on the arc AB.
2. The angle at the centre is double the angle at the circumference.
3. So x = 140 ÷ 2 = 70°.

Answer: 70°



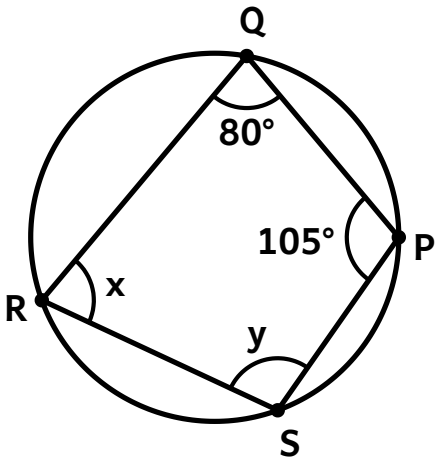
Find the angle x.

WORKED EXAMPLE

AB is a diameter and C is on the circle. Angle CAB is 38°. Find angle ABC.

1. AB is a diameter, so the angle in a semicircle at C is 90°.
2. The angles of triangle ABC add up to 180°.
3. Angle ABC = 180 − 90 − 38 = 52°.

Answer: 52°



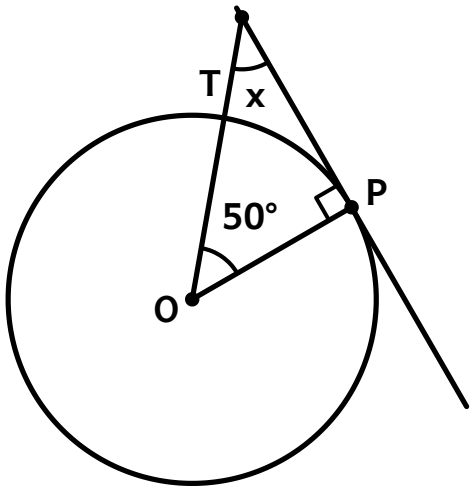
PQRS is a cyclic quadrilateral. Find x and y.

WORKED EXAMPLE

PQRS is a cyclic quadrilateral. Angle P is 105° and angle Q is 80°. Find angle R and angle S.

1. Opposite angles of a cyclic quadrilateral add up to 180°.
2. Angle R is opposite angle P, so R = 180 − 105 = 75°.
3. Angle S is opposite angle Q, so S = 180 − 80 = 100°.
4. Check: the four angles are 105°, 80°, 75° and 100°. They add up to 360°.

Answer: R is 75° and S is 100°



PT is a tangent. Find the angle x.

WORKED EXAMPLE

PT is a tangent to a circle with centre O. Angle POT is 50°. Find angle OTP.

1. The tangent PT meets the radius OP at 90°. So angle OPT = 90°.
2. The angles of triangle OPT add up to 180°.
3. Angle OTP = 180 − 90 − 50 = 40°.
4. Check: 90° + 50° + 40° = 180°.

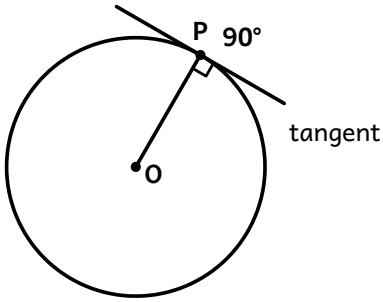
Answer: 40°

SEE IT THREE WAYS

With things

Stand a coin up against the straight edge of a ruler. The ruler touches the coin at one point. Use the corner of a sheet of paper to check. The line from the middle of the coin to that point is square to the ruler.

As a picture



In symbols

The angle between a tangent and the radius to the touching point is 90°.

TRY IT

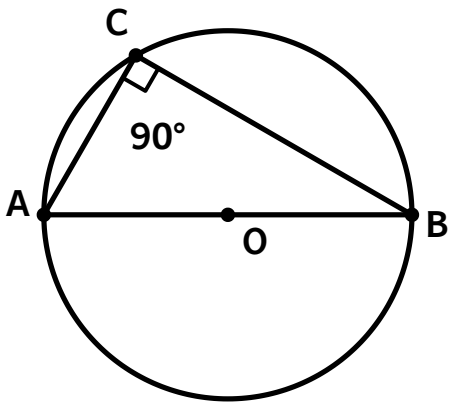
Check the angle in a semicircle

You need: a round tin lid or a compass, a ruler, the corner of a sheet of paper, a pencil.

1. Draw round the tin lid, or use a compass. Mark the centre and draw a diameter through it.
2. Mark any point C on the circle. Join C to both ends of the diameter.
3. Fit the corner of a sheet of paper into the angle at C.
4. Move C to a new place on the circle and try again.

What you will see: The corner fits exactly each time. The angle in a semicircle is 90° wherever C is.

If you cannot try it, this is what happens.



A WRONG IDEA MANY PEOPLE HAVE

~~The angle at the centre equals the angle at the circumference on the same arc.~~

Why it feels right: Both angles are drawn from the same two points, so they look as if they should match.

The test that breaks it: Take a diameter AB. It makes a straight angle of 180° at the centre. The angle at any point C on the circle is 90°, as the tin lid test showed. They are not equal.

What is true: The angle at the centre is double the angle at the circumference on the same arc.

KEY WORDS

chord
a straight line joining two points on a circle.

stand on
an angle stands on the arc between the ends of its arms, on the side away from the angle.

semicircle
half of a circle, cut off by a diameter.

cyclic quadrilateral
a four sided shape with all four corners on a circle.

tangent
a straight line that touches a circle at one point.

BACK TO YOUR GUESS

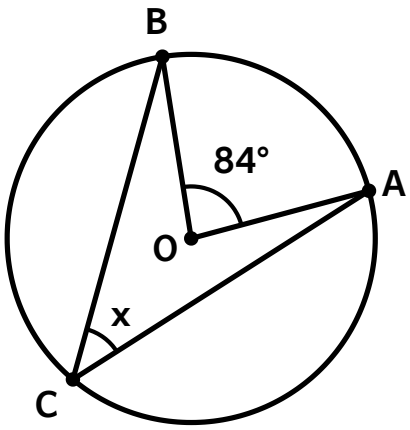
The answer is B: 50°, half of the angle at the centre.

The angle at the edge, on the same arc, is half the angle at the centre. So 100° at the centre makes 50° at the edge. This chapter shows why, and uses it to find angles.

CHECK YOURSELF

Try each one before you look. A wrong answer tells you where to go next.

1. The angle at the centre is 84°. What is the angle x at the circumference on the same arc?



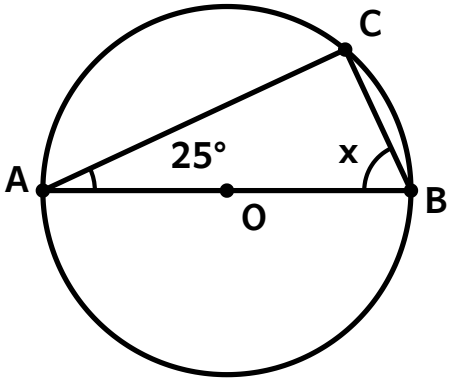
A 84°

B 42°

C 168°

Answer at the back of the book.

2. AB is a diameter and C is on the circle. Angle CAB is 25°. What is angle ABC?



A 90°

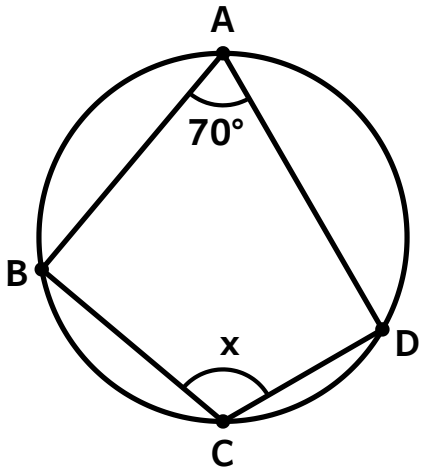
B 155°

C 65°

D 25°

Answer at the back of the book.

3. ABCD is a cyclic quadrilateral. Angle A is 70°. What is angle C?



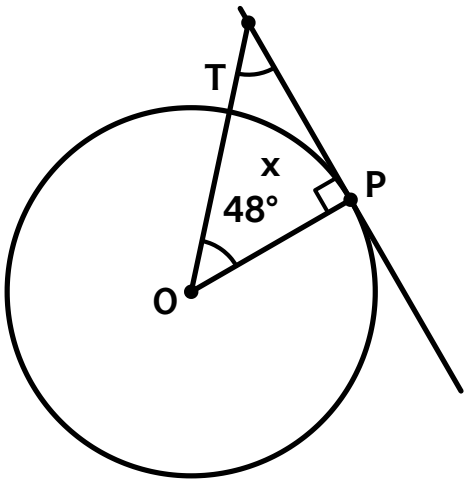
A 110°

B 70°

C 290°

Answer at the back of the book.

4. PT is a tangent to a circle with centre O. Angle POT is 48°. What is angle OTP?



A 132°

B 48°

C 42°

Answer at the back of the book.

5. Which statement about a circle is true?

   A The angle at the centre equals the angle at the circumference on the same arc.

   B The angle at the centre is double the angle at the circumference on the same arc.

   C Opposite angles of a cyclic quadrilateral are equal.

   Answer at the back of the book.

6. **From an earlier chapter**

   A sector has a radius of 8 cm and an angle of 45°. How long is its arc, in centimetres, to 1 decimal place?

   Answer at the back of the book.

7. The angle at the centre of a circle is 120°. What is the angle at the circumference on the same arc, in degrees?

   Answer at the back of the book.

8. ABCD is a cyclic quadrilateral. Angle B is 83°. What is angle D, in degrees?

   Answer at the back of the book.

9. A tangent touches a circle at P. The centre is O and T is a point on the tangent. Angle OTP is 31°. What is angle POT, in degrees?

   Answer at the back of the book.

KEEP IT

Copy this into your exercise book.

Angle at the centre = 2 × angle at the circumference, same arc. Angle in a semicircle = 90°. Cyclic quadrilateral: opposite angles add to 180°. A tangent is square to the radius: 90°.

Draw: A circle with a chord and the angles at the centre and the edge. Add a tangent with a right angle mark.

Key words, A to Z

Every key word in this book, what it means, and the chapter that teaches it.

anticlockwise
in the opposite direction to the hands of a clock. M4.4.1

arc
a part of the circumference of a circle. M4.4.7

area scale factor
the number every area is multiplied by. It is k^2, where k is the length scale factor. M4.4.6

ASA
two angles and the side between them are equal. If the side is not between them, the test is called AAS and it works too. M4.4.4

centre of enlargement
the point the distances are measured from. It does not move. M4.4.3

centre of rotation
the point that stays still while the shape turns. M4.4.1

chord
a straight line joining two points on a circle. M4.4.8

clockwise
in the direction the hands of a clock move. M4.4.1

column vector
two numbers in brackets, one above the other. The top number is the move right and the bottom number is the move up. M4.4.2

combined move
two moves one after the other. For two translations it is one translation. M4.4.2

congruent
having exactly the same size and form. One shape fits exactly on the other. M4.4.4

corresponding
matching. Corresponding sides and angles sit in the same place in each shape. M4.4.4

cyclic quadrilateral
a four sided shape with all four corners on a circle. M4.4.8

enlargement
a transformation that multiplies every distance from a centre by the same number. The shape keeps its form. M4.4.3

image
the shape that a transformation makes. M4.4.1

mirror line
the line a shape is flipped over. A point and its image are the same distance from it. M4.4.1

proof
a chain of reasons showing a fact must be true. M4.4.4

reflection
a flip of a shape over a mirror line. M4.4.1

RHS
right angle, hypotenuse and one other side are equal. M4.4.4

rotation
a turn of a shape about a fixed point. M4.4.1

SAS
two sides and the angle between them are equal. M4.4.4

scale factor
the number every length is multiplied by. A scale factor between 0 and 1 makes the shape smaller. M4.4.3

sector
a slice of a circle between two radii. It has an arc for its curved edge. M4.4.7

semicircle
half of a circle, cut off by a diameter. M4.4.8

similar
having the same form, with equal matching angles and matching sides in the same ratio. One shape is an enlargement of the other. M4.4.5

SSS
three sides of one triangle equal three sides of another. M4.4.4

stand on
an angle stands on the arc between the ends of its arms, on the side away from the angle. M4.4.8

tangent
a straight line that touches a circle at one point. M4.4.8

transformation

a rule that moves *a* shape: *a* flip, *a* turn, *a* slide or *an* enlargement. M4.4.1

translation

a slide of *a* shape. Every point moves the same distance in the same direction. M4.4.2

volume scale factor

the number every volume is multiplied by. It is k^3, where k is the length scale factor. M4.4.6

Answers

M4.4.1. Reflections and rotations

1. **(5, 2)**

   The mirror line y = x swaps the two numbers of every point. The corner (2, 5) goes to (5, 2).

   - **If you chose A:** You changed the sign of x. That is a reflection in the y axis. A reflection in y = x swaps the two numbers. Go to M4.4.1.
   - **If you chose C:** You changed the sign of y. That is a reflection in the x axis. A reflection in y = x swaps the two numbers. Go to M3.9.3.

2. **(5, 2)**

   The corner (1, 2) is 2 units left of the line x = 3. Its image is 2 units right of the line, at (5, 2). The y number stays the same.

   - **If you chose A:** That is the reflection in the y axis, the line x = 0. The mirror here is the line x = 3. Go to M4.4.1.
   - **If you chose B:** The image is not on the mirror line. It is as far past the line as the corner is before it, which is 2 units. Go to M4.4.1.

3. **(1, −4)**

   A 90° clockwise turn about O sends (x, y) to (y, −x). So (4, 1) goes to (1, −4).

   - **If you chose B:** You turned it the other way, anticlockwise. Clockwise is the way the hands of a clock move. Go to M4.4.1.
   - **If you chose C:** That is a half turn, 180°. A quarter turn of 90° is half as much. Go to M4.4.1.

4. **90° anticlockwise about O**

   A turn of 90° anticlockwise about O sends (x, y) to (−y, x). The corner (1, 1) goes to (−1, 1) and (3, 1) goes to (−1, 3). Those are corners of B.

   - **If you chose A:** Clockwise would put the image below the x axis. B is above it, so the turn is the other way. Go to M4.4.1.
   - **If you chose B:** A half turn would put B below and to the left of O, in the opposite corner of the grid. Go to M4.4.1.
   - **If you chose D:** The angle and the direction are right, but the centre is not. Every point of A is turned about O, and O stays still. Go to M4.4.1.

5. **90° clockwise about (2, 1)**

   A rotation needs a centre, an angle and a direction. Only the last choice gives all three.

   - **If you chose A:** This gives the angle and direction but not the centre. Different centres give different images. Go to M4.4.1.
   - **If you chose B:** This gives the centre but not the angle or the direction. Go to M4.4.1.
   - **If you chose C:** This gives the angle and the centre but not the direction. A 90° turn can go either way. Go to M4.4.1.

6. **2**

   A reflection in y = x swaps the two numbers. (7, 2) goes to (2, 7), so the x number is 2.

7. **5**

   The corner is 2 units left of the line x = 3. The image is 2 units right of it, so its x number is 3 + 2 = 5.

8. **6**

   A half turn changes right into left and up into down. The image is 4 units left of C and 6 units below it.

9. **2**

   A 90° clockwise turn about O sends (x, y) to (y, −x). So (2, 6) goes to (6, −2), which is 2 units below the x axis.

M4.4.2. Translations and combined moves

1. **top number 4, bottom number −2**

   The corner (1, 4) moves to (5, 2). That is 4 right and 2 down, so the top number is 4 and the bottom number is −2.

   - **If you chose A:** You wrote the numbers in the wrong order. The top number is the move right, which is 4. The bottom number is the move up or down. Go to M4.4.2.
   - **If you chose C:** You described the move from B back to A. The vector must go from the shape to its image. Go to M4.4.2.
   - **If you chose D:** The shape moves down, not up. A move down needs a minus sign on the bottom number. Go to M4.4.2.

2. **(6, 2)**

   Add 4 to x: 2 + 4 = 6. Add 3 to y: −1 + 3 = 2. The corner goes to (6, 2).

   - **If you chose A:** You added the numbers the wrong way round. The top number, 4, goes with x. The bottom number, 3, goes with y. Go to M4.4.2.
   - **If you chose B:** A vector says how far to move, not where to land. Add 4 to the x number and 3 to the y number of the corner. Go to M4.4.2.
   - **If you chose D:** You took the vector away. A translation by the vector adds its numbers. Go to M4.4.2.

3. **−3 on top, 3 below**

   Add the top numbers: 2 + (−5) = −3. Add the bottom numbers: 1 + 2 = 3.

   - **If you chose B:** You took the second move away from the first. Two moves in a row are added. Go to M4.4.2.
   - **If you chose C:** You multiplied. Two moves in a row are added, not multiplied. Go to M4.4.2.

4. **−5 on top, 0 below**

   The top number is the move right, and left is the minus direction, so it is −5. The bottom number is 0, because there is no move up or down.

   - **If you chose A:** A move up or down goes on the bottom. This shape moves sideways, so the 5 belongs on top. Go to M4.4.2.
   - **If you chose B:** A positive top number is a move right. A move left needs a minus sign. Go to M4.4.2.

5. **(3, −2)**

   From (1, 1) to (4, −1) is 3 right and 2 down, so the vector has 3 on top and −2 below. Every point moves by that vector, so (0, 0) goes to (3, −2).

   - **If you chose A:** That is where (1, 1) goes. A translation moves each point by the same vector, so (0, 0) lands somewhere else. Go to M4.4.2.
   - **If you chose B:** You used the vector the wrong way round. The vector goes from (1, 1) to (4, −1), so it is added, not taken away. Go to M4.4.2.
   - **If you chose C:** A translation moves every point, and (0, 0) is not a special point. The origin moves with the rest. Go to M4.4.2.

6. **4**

   A 90° anticlockwise turn about (0, 0) sends (x, y) to (−y, x). So (4, 1) goes to (−1, 4), and the y number is 4.

7. **7**

   Add the bottom number to y: 2 + 5 = 7.

8. **7**

   The move up is the new y minus the old y: from 2 to 9 is 7.

9. **9**

   Add the bottom numbers: 5 + 4 = 9.

M4.4.3. Enlargements

1. **(7, 4)**

   The corner (3, 2) is 2 right and 1 up from C. Multiply by 3: 6 right and 3 up. Counting from C at (1, 1) gives (7, 4).

   - **If you chose A:** You added 3 to each number. An enlargement multiplies the distances from the centre by the scale factor. Go to M4.4.3.
   - **If you chose C:** You multiplied only the right step. Both numbers of the step from the centre are multiplied. Go to M4.4.3.

2. **(4, 3)**

   The corner (6, 4) is 4 right and 2 up from C. Half of that is 2 right and 1 up. Counting from C at (2, 2) gives (4, 3).

   - **If you chose A:** You halved the numbers of the corner itself. Halve the step from the centre instead, then count from the centre. Go to M4.4.3.
   - **If you chose B:** You doubled the step from the centre. A scale factor of 1/2 halves it. Go to M4.4.3.

3. **The side is 12 cm and the angles stay the same.**

   Lengths are multiplied by the scale factor: 4 × 3 = 12 cm. Angles do not change.

   - **If you chose A:** You added 3 cm. An enlargement multiplies each length by the scale factor. Go to M4.4.3.
   - **If you chose B:** The side is right, but angles do not change in an enlargement. The shape keeps its form. Go to M4.4.3.

4. **(2, 1)**

   The line through (3, 2) and (4, 3) and the line through (4, 2) and (6, 3) cross at (2, 1). That is the centre.

   - **If you chose A:** This is a corner of A. The centre is where the lines through matching corners meet, and it is not a corner of A. Go to M4.4.3.
   - **If you chose B:** This is a corner of A′. The centre is where the lines through matching corners meet. Go to M4.4.3.
   - **If you chose C:** Draw the line through (3, 2) and (4, 3) and the line through (4, 2) and (6, 3). They do not cross at this point. Go to M4.4.3.

5. **5 cm**

   Multiply the length by the scale factor: 10 × 1/2 = 5 cm.

   - **If you chose B:** You doubled the length. A scale factor of 1/2 is half, so the image is smaller than the original. Go to M4.4.3.
   - **If you chose C:** You took 1/2 away. Scale factors multiply. Half of 10 cm is 5 cm. Go to M4.4.3.

6. **5**

   Add the bottom number to y: 3 + 2 = 5.

7. **15**

   The corner is 2 right and 5 up from O. Multiply by 3: 6 right and 15 up. The y number is 15.

8. **9**

   The corner is 4 right of C. Double it: 8 right of C. Counting from x = 1 gives 1 + 8 = 9.

9. **3**

   The scale factor is the new length divided by the old length: 15 ÷ 5 = 3.

M4.4.4. Congruent triangles

1. **SSS**

   All three pairs of sides are equal, and that is the SSS test.

   - **If you chose A:** SAS needs an angle between two of the sides. No angle is marked, only the three sides. Go to M4.4.4.
   - **If you chose C:** RHS is for right angled triangles. No right angle is marked here. Go to M4.4.4.

2. **No. Their third sides are different lengths. SSA cannot prove two triangles congruent.**

   The equal angle is not between the two equal sides. That is SSA, which is not a test of congruence. The picture shows two different triangles with different third sides.

   - **If you chose A:** The angle of 40° is not between the 5 cm and 7 cm sides. SAS needs the angle between the two sides. Go to M4.4.4.
   - **If you chose B:** SSS needs all three sides equal. The third sides are different lengths in the picture. Go to M4.4.4.
   - **If you chose C:** Triangles that look alike can differ in size. A reason must be one of the four tests. Go to M4.4.4.

3. **RHS**

   The triangles each have a right angle, equal hypotenuses and one more equal side. That is RHS.

   - **If you chose A:** ASA needs two equal angles and a side. Here one angle is marked, and there are two sides. Go to M4.4.4.
   - **If you chose B:** SAS needs the angle between the two equal sides. The right angle is not between the hypotenuse and the 3 cm side. Go to M4.4.4.

4. **ASA**

   Two angles are equal and the side between them is equal. That is ASA.

   - **If you chose B:** SSS needs three equal sides. Only one side is marked here, with two angles. Go to M4.4.4.
   - **If you chose C:** RHS needs a right angle. None is marked here. Go to M4.4.4.

5. **5 cm**

   M is the middle of BC, so BM is half of BC. Half of 10 cm is 5 cm.

   - **If you chose A:** That is the length of AB. BM is part of BC, the base of the triangle. Go to M4.4.4.
   - **If you chose B:** That is the whole base BC. M is the middle of BC, so BM is only half of it. Go to M4.4.4.

6. **24**

   An enlargement multiplies each length by the scale factor: 6 × 4 = 24 cm.

7. **5**

   Congruent triangles have the same three side lengths. DEF has 6 cm and 9 cm, so the third side matches the 5 cm of ABC.

8. **40**

   The angles of a triangle add up to 180°. Angle A = 180° − 70° − 70° = 40°.

9. **70**

   The angles of a triangle add up to 180°. The third angle is 180° − 50° − 60° = 70° in both triangles. So all three angles match, and with the equal side the triangles are congruent by AAS.

M4.4.5. Similar shapes

1. **Yes. The missing angles are 70° and 50°, so all three angles match.**

   The first triangle has a third angle of 180° − 50° − 60° = 70°. The second has 180° − 60° − 70° = 50°. Both triangles have the angles 50°, 60° and 70°, so they are similar.

   - **If you chose B:** You compared only the marked angles. Work out the third angle of each triangle too. The three angles of a triangle add up to 180°. Go to M4.4.5.
   - **If you chose C:** Not every pair of triangles is similar. The angles must match, and being triangles is not enough. Go to M4.4.5.

2. **15 cm**

   The scale factor is 10 ÷ 4 = 2.5. Multiply BC by it: 6 × 2.5 = 15 cm.

   - **If you chose A:** You added. DE is 6 cm more than AB, so you added 6 to BC. Similar shapes grow by multiplying by the scale factor. Go to M4.4.5.
   - **If you chose B:** You used the scale factor the wrong way round. DEF is the larger triangle, so EF must be longer than BC. Go to M4.4.5.

3. **12 cm**

   The scale factor from ABC to PQR is 8 ÷ 12 = 2/3. Multiply BC by it: 18 × 2/3 = 12 cm.

   - **If you chose A:** You used the scale factor from the small triangle to the large one. PQR is smaller, so QR is shorter than BC. Go to M4.4.5.
   - **If you chose C:** You took 4 cm away because 8 is 4 less than 12. Similar shapes shrink by multiplying, not by taking away. Go to M4.4.5.

4. **Two squares**

   Every square has four right angles and four equal sides. Any two squares have equal angles, and their sides are in the same ratio, so they are similar.

   - **If you chose A:** Two rectangles have equal angles, but their sides need not be in the same ratio. A 2 cm by 4 cm rectangle and a 2 cm by 8 cm rectangle are not similar. Go to M4.4.5.
   - **If you chose B:** One equal angle is not enough. The other angles can differ, so the triangles can have different forms. Go to M4.4.5.
   - **If you chose D:** Kites can be long and thin or short and fat. Two kites need not have equal angles or sides in the same ratio. Go to M4.4.5.

5. **16 m**

   The scale factor is 24 ÷ 3 = 8. The building is 8 times as tall as the stick: 2 × 8 = 16 m.

   - **If you chose A:** You used the scale factor the wrong way round, 3 ÷ 2. The building shadow is 8 times the stick shadow, so use 24 ÷ 3. Go to M4.4.5.
   - **If you chose C:** You added. The building shadow is 21 m longer, and you added 21 to the stick. Multiply by the scale factor instead. Go to M4.4.5.

6. **50**

   The base angles of an isosceles triangle are equal, so angle C is also 65°. Angle A = 180° − 65° − 65° = 50°.

7. **21**

   The scale factor is 15 ÷ 5 = 3. Multiply 7 by it: 7 × 3 = 21 cm.

8. **5**

   The scale factor is 4 ÷ 12 = 1/3. Multiply 15 by it: 15 ÷ 3 = 5 cm.

9. **20**

   An enlarged photo is similar to the original. The scale factor is 30 ÷ 12 = 2.5. Multiply 8 by it: 8 × 2.5 = 20 cm.

M4.4.6. Similar areas and volumes

1. **1 080 cm^3**

   The volume scale factor is $k^3 = 3 \times 3 \times 3 = 27$. The large box has a volume of $40 \times 27 = 1\,080$ cm^3.

   - **If you chose A:** You multiplied the volume by 3. A volume has three lengths, so it is multiplied by 3 × 3 × 3. Go to M4.4.6.
   - **If you chose B:** You multiplied by 3 × 3 = 9, which is for an area. A volume needs three lengths, so use 3 × 3 × 3. Go to M4.4.6.

2. **800 cm^2**

   The area scale factor is $k^2 = 4 \times 4 = 16$. The new area is 50 × 16 = 800 cm^2.

   - **If you chose A:** You multiplied the area by 4, the length scale factor. An area has two lengths, so multiply by 4 × 4. Go to M4.4.6.
   - **If you chose C:** You multiplied by 4 × 4 × 4, which is for a volume. An area has only two lengths, so use 4 × 4. Go to M4.4.6.

3. **3**

   The area scale factor is 108 ÷ 12 = 9. The length scale factor is the square root of 9, which is 3.

   - **If you chose B:** You divided 108 by 12. That is the area scale factor. The length scale factor is its square root. Go to M4.4.6.
   - **If you chose C:** You took 12 away from 108. Scale factors are found by dividing, and then a square root is needed for lengths. Go to M4.4.6.

4. **2**

   The volume scale factor is 800 ÷ 100 = 8. The length scale factor k has $k^3 = 8$, so k = 2, because 2 × 2 × 2 = 8.

   - **If you chose A:** You divided 800 by 100. That is the volume scale factor. The length scale factor is its cube root. Go to M4.4.6.
   - **If you chose B:** You halved 8. Volume scales by k × k × k, so you need the number k with k × k × k = 8. Go to M4.4.6.

5. **The *area* is 25 times *as* big.**

   The *area* scale factor is $k^2 = 5 \times 5 = 25$.

   - **If you chose A:** Five times is the scale factor for lengths. *Area* has two lengths, so it grows by more than that. Go to M4.4.6.
   - **If you chose C:** You added 5 and 5. *Areas* are found by multiplying, so the factor is 5 × 5. Go to M4.4.6.

6. **15**

   The scale factor is 12 ÷ 4 = 3. Multiply 5 by it: 5 × 3 = 15 cm.

7. **54**

   The *area* scale factor is 3 × 3 = 9. The new *area* is 6 × 9 = 54 cm^2.

8. **120**

   The volume scale factor is 2 × 2 × 2 = 8. The larger jar holds 15 × 8 = 120 ml.

9. **3**

   The volume scale factor is 270 ÷ 10 = 27. The length scale factor k has k × k × k = 27, so k = 3.

M4.4.7. Arcs and sectors

1. **9.4 cm**

   To 1 decimal place, the circumference is $2 \times \pi \times 6 = 37.7$ cm. The sector is 90 out of 360, so the arc is $90 \div 360 \times 37.7 = 9.4$ cm.

   - **If you chose A:** You found the whole circumference. The arc is only a part of it, 90 out of 360. Go to M4.4.7.
   - **If you chose C:** You used the area rule, πr^2, and took 90 out of 360 of it. That is the area of the sector. An arc is a length, so use the circumference, $2\pi r$. Go to M4.4.7.

2. **28.3 cm^2**

   The area of the whole circle is $\pi \times 9 \times 9 = 254.5$ cm^2. The sector is 40 out of 360, so its area is $40 \div 360 \times 254.5 = 28.3$ cm^2.

   - **If you chose A:** You found the area of the whole circle. The sector is only 40 out of 360 of it. Go to M4.4.7.
   - **If you chose B:** You used the arc length rule, with $2\pi r$. An area uses πr^2, the area of the whole circle. Go to M4.4.7.

3. **17.9 cm**

   The arc is $90 \div 360 \times 2 \times \pi \times 5 = 7.9$ cm. The two radii are $2 \times 5 = 10$ cm. The perimeter is $7.9 + 10 = 17.9$ cm.

   - **If you chose B:** You gave only the arc length. The perimeter goes all the way round, so add the two straight edges. Go to M4.4.7.
   - **If you chose C:** You added one radius. A sector has two straight edges, and both are radii. Add 2×5. Go to M4.4.7.

4. **Multiply the fraction of the circle by the circumference.**

   An arc is a part of the circumference. The fraction of the circle is the angle ÷ 360, and the circumference is $2\pi r$.

   - **If you chose A:** Adding the angle to the radius mixes degrees and centimetres. An arc is a fraction of the circumference. Go to M4.4.7.
   - **If you chose B:** This does not give a length in centimetres. Degrees must first be turned into a fraction of 360. Go to M4.4.7.
   - **If you chose C:** This gives the area of a sector. An arc is a length, so it uses the circumference instead. Go to M4.4.7.

5. **12.6 m^2**

   The whole circle has an area of $\pi \times 4 \times 4 = 50.3$ m^2. The sector is 90 ÷ 360 of it, which is 12.6 m^2.

   - **If you chose A:** You found the area of the whole circle. A sector with an angle of 90° is only a quarter of it. Go to M4.4.7.
   - **If you chose C:** The number is right, but the unit is not. An area is in square metres, m^2, and not in metres. Go to M4.4.7.

6. **63**

   The area scale factor is $3 \times 3 = 9$. The new area is $7 \times 9 = 63$ cm^2.

7. **18.8**

   The circumference is $2 \times \pi \times 12 = 75.4$ cm. The sector is 90 ÷ 360 of it, so the arc is 18.8 cm.

8. **47.1**

   The area of the whole circle is $\pi \times 6 \times 6 = 113.1$ cm^2. The sector is 150 ÷ 360 of it, so its area is 47.1 cm^2.

9. **22.3**

   To 1 decimal place, the arc is $45 \div 360 \times 2 \times \pi \times 8 = 6.3$ cm. The two radii are $2 \times 8 = 16$ cm. The perimeter is $6.3 + 16 = 22.3$ cm.

M4.4.8. Angle rules in circles

1. **42°**

   The angle at the centre is double the angle at the circumference on the same arc. So $x = 84 \div 2 = 42°$.

   - **If you chose A:** The angle at the circumference is not the same as the angle at the centre. It is half as big. Go to M4.4.8.
   - **If you chose C:** You doubled the angle. It is the angle at the centre that is double, so the angle at the circumference is half. Go to M4.4.8.

2. **65°**

   The angle in a semicircle at C is 90°. The angles of the triangle add up to 180°, so angle ABC = 180 − 90 − 25 = 65°.

   - **If you chose A:** The angle at C is 90°, but the question asks for the angle at B. Use the angles of the triangle. Go to M4.4.8.
   - **If you chose B:** You took 25° away from 180°. The triangle ABC also has a right angle at C, which must be taken away too. Go to M4.4.8.
   - **If you chose D:** You copied the angle at A. The two angles are not equal, because triangle ABC is not isosceles. Go to M4.4.8.

3. **110°**

   Opposite angles of a cyclic quadrilateral add up to 180°. Angle C = 180 − 70 = 110°.

   - **If you chose B:** Opposite angles of a cyclic quadrilateral add up to 180°. They are not equal unless each one is 90°. Go to M4.4.8.
   - **If you chose C:** You took 70° away from 360°. That rule is for all four angles together. For two opposite angles the total is 180°. Go to M4.4.8.

4. **42°**

   The tangent meets the radius at 90°, so angle OPT = 90°. The angles of triangle OPT add up to 180°, so angle OTP = 180 − 90 − 48 = 42°.

   - **If you chose A:** You took 48° away from 180° and forgot the right angle at P. The angles of triangle OPT include the 90°. Go to M4.4.8.
   - **If you chose B:** You copied the angle at O. The tangent makes a right angle with the radius, so the other angle is not equal. Go to M4.4.8.

5. **The angle at the centre is double the angle at the circumference on the same arc.**

   This is the first rule of the chapter. The angle at the centre is double the angle at the circumference when both stand on the same arc.

   - **If you chose A:** They are not equal. The angle at the centre is double the angle at the circumference on the same arc. Go to M4.4.8.
   - **If you chose C:** Opposite angles add up to 180°. They are equal only when each is 90°. Go to M4.4.8.

6. **6.3**

   The circumference is 2 × π × 8 = 50.3 cm. The sector is 45 out of 360 of it, so the arc is 6.3 cm.

7. **60**

   The angle at the circumference is half the angle at the centre: 120 ÷ 2 = 60°.

8. **97**

   Opposite angles of a cyclic quadrilateral add up to 180°. Angle D = 180 − 83 = 97°.

9. **59**

   The tangent meets the radius at 90°, so angle OPT = 90°. The angles of triangle OPT add up to 180°, so angle POT = 180 − 90 − 31 = 59°.

About this book

For learners who can reflect a shape in an axis on a grid, find angles in triangles and work out the area of a circle. In many schools this fits teenagers of about 14 to 16, and adults who want to refresh transformations and circle rules can start here too. Each chapter takes one or two sessions. A learner needs squared paper, a ruler and a pencil. A learner can work alone: every exercise has its answer and the reason for it.

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