●●●●○ Level 4 · Mathematics M4.6

Probability

Work out the chances of one or more events, and use tables, trees and diagrams to keep track.

List the outcomes of two events in a table and find a probability from it. Estimate a probability from an experiment, and use the OR rule and the AND rule. Draw tree diagrams, with and without putting back, and use Venn diagrams, sets and two way tables to find the chance of one event given another. End with counting choices.

YOU ARE READY FOR THIS BOOK IF

You can find the probability of rolling an even number on a dice.

One step up	M5.7 Probability distributions and inference
This book	**M4.6 Probability**
One step down	M3.7 Statistics and chance

Contents

M4.6.1 Listing all the outcomes

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M4.6.5 Picking without putting back

M4.6.6 Venn diagrams and sets

M4.6.7 Conditional probability

M4.6.8 Counting choices

Key words, A to Z

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M4.6.1 CHAPTER 1

Listing all the outcomes

I roll two dice and add the scores. Which total comes up most often?

YOUR GUESS

Pick one before you read on. It is fine to be wrong.

- ☐ **A** All the totals come up equally often, because each total is one outcome.
- ☐ **B** A total of 7, because more pairs of scores make it.
- ☐ **C** A total of 12, because it is the biggest.

IF THIS FEELS HARD, READ THIS FIRST

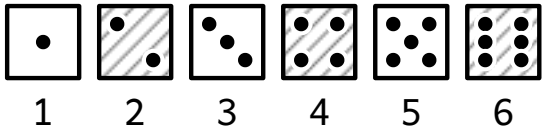
- Level 3, Statistics and chance, M3.7.7. If finding a probability from equally likely outcomes looks new, read this first.

BY THE END YOU CAN

- List all the outcomes of two events in a table
- Find a probability from a table of outcomes

An **outcome** is one possible result, such as a 4 on a dice. An **event** is the result you are interested in, such as "an even score". It can hold more than one outcome.

When every outcome is **equally likely**, the probability of an event is a count divided by a count. It is the number of outcomes in the event, divided by the number of outcomes in all. For an even score on a dice, that is $^3/_6$, which is $^1/_2$.



even scores

Three of the six equally likely scores are even.

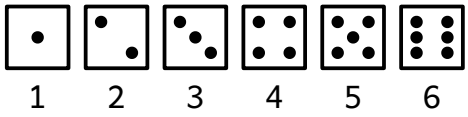
Now two things happen together: a coin is tossed and a dice is rolled. To count the outcomes without missing any, list them in a table. Put the coin outcomes down the side and the dice scores across the top.

DRAW IT FIRST

List every outcome of a coin and a dice

coin (H) (T)

dice



1 List the outcomes of each thing. A coin has two: H for head and T for tail. A dice has six: the scores 1 to 6.

dice

coin	1	2	3	4	5	6
H	H1	H2	H3	H4	H5	H6
T	T1	T2	T3	T4	T5	T6

2 Make a table. Every coin outcome meets every score in one cell. Hatch the cells that fit the event: a head and an even score.

dice

coin	1	2	3	4	5	6
H	H1	H2	H3	H4	H5	H6
T	T1	T2	T3	T4	T5	T6

12 outcomes. The 3 hatched cells are a head and an even score.

WORKED EXAMPLE

A coin is tossed and a dice is rolled. What is the probability of a head and an even score?

1. The coin has 2 outcomes and the dice has 6. Each coin outcome goes with each score.
2. So there are 2 × 6 = 12 outcomes in all, and they are equally likely.
3. The cells with a head and an even score are H2, H4 and H6. That is 3 outcomes.
4. P(head and even) = 3/12 = 1/4.

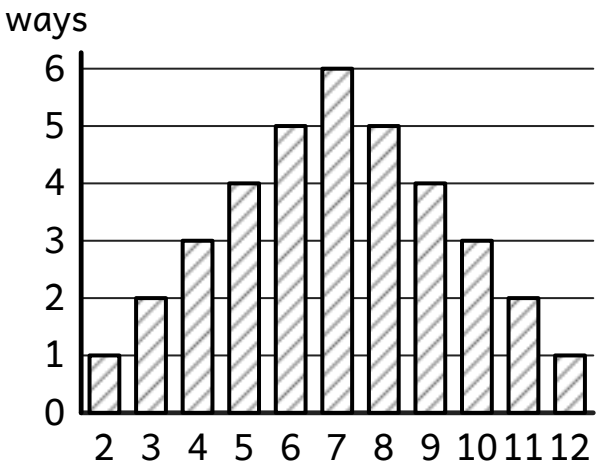
Answer: 1/4

The same table works for two dice. The first dice goes across the top and the second dice goes down the side. Each cell shows the total of its two scores. There are 36 cells.

	dice 1					
dice 2	**1**	**2**	**3**	**4**	**5**	**6**
1	2	3	4	5	6	7
2	3	4	5	6	7	8
3	4	5	6	7	8	9
4	5	6	7	8	9	10
5	6	7	8	9	10	11
6	7	8	9	10	11	12

A total of 7 appears in 6 of the 36 cells.

The 36 pairs are equally likely, because each dice is fair and one dice cannot change the other. But the 11 totals from 2 to 12 are not equally likely. A total of 7 fills 6 cells. A total of 2 fills only one cell.



The number of ways to make each total. The most likely total is 7.

WORKED EXAMPLE

Two dice are rolled and added. What is the probability that the total is 10 or more?

1. Use the table of 36 equally likely pairs.
2. A total of 10 appears 3 times, 11 appears 2 times and 12 appears once.
3. That is 3 + 2 + 1 = 6 cells.
4. P(10 or more) = 6/36 = 1/6.

Answer: 1/6

total	ways
10	3
11	2
12	1
10 or more	6

Three totals make 10 or more, and together they fill 6 cells.

TRY IT

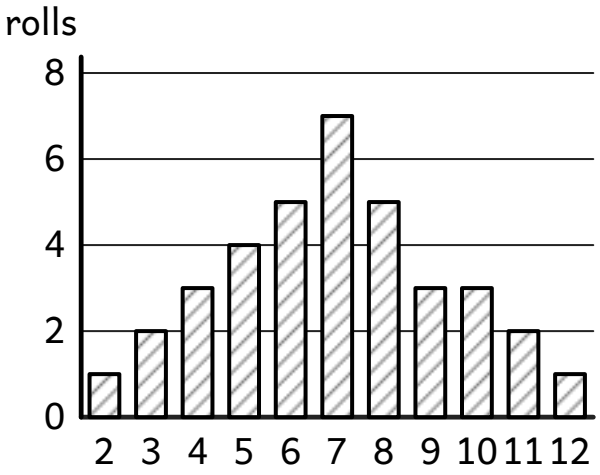
Roll two dice 36 times

You need: two dice, or two sets of six pieces of paper numbered 1 to 6, each set in its own bag, a pencil and paper for a tally.

1. Roll both dice and add the scores. Make a mark in the tally against that total.
2. Do this 36 times.
3. Draw a bar for each total.
4. Compare with the bar chart of ways. Do not expect them to match exactly.

What you will see: The middle totals, 6, 7 and 8, come up most often. The totals 2 and 12 come up least. Your bars will not match the ways exactly, because the rolls are chance.

If you cannot try it, this is what happens.



A WRONG IDEA MANY PEOPLE HAVE

~~There are 11 totals, so each total has a probability of 1/11.~~

Why it feels right: The totals 2 to 12 are 11 results, and it seems fair to give each one the same share.

The test that breaks it: List the pairs. A total of 12 needs a 6 and a 6, so it is 1 of 36. A total of 7 can be made six ways, so it is 6 of 36.

What is true: Count only outcomes that are equally likely. The 36 pairs are. The 11 totals are not.

KEY WORDS

outcome
one possible result, such as a 5 on a dice or a tail on a coin.

event
the result you are interested in. It is made of one or more outcomes.

equally likely
having the same chance of happening, like the six faces of a fair dice.

probability
a number from 0 to 1 that measures how likely an event is.

BACK TO YOUR GUESS

The answer is B: A total of 7, because more pairs of scores make it.

There are 36 pairs of scores, and all 36 are equally likely. A total of 7 can be made 6 ways. A total of 2 can be made only 1 way, with a 1 and a 1. So 7 comes up most often.

CHECK YOURSELF

Try each one before you look. A wrong answer tells you where to go next.

1. A coin is tossed and a spinner with 4 equal sections, numbered 1 to 4, is spun. How many outcomes are there in all?

   A 6

   B 8

   C 4

   Answer at the back of the book.

2. A coin is tossed and a dice is rolled. What is the probability of a tail and a 5?

   A 1/6

   B 1/2

   C 1/12

   Answer at the back of the book.

3. Two dice are rolled and added. Use the table. What is the probability that the total is 8?

		dice 1				
dice 2	**1**	**2**	**3**	**4**	**5**	**6**
1	2	3	4	5	6	7
2	3	4	5	6	7	8
3	4	5	6	7	8	9
4	5	6	7	8	9	10
5	6	7	8	9	10	11
6	7	8	9	10	11	12

   A 5/36

   B 1/11

   C 8/36

   Answer at the back of the book.

4. Two coins are tossed. Which list shows every outcome?

   A HH, TT

   B HH, HT, TT

   C H, T

   D HH, HT, TH, TT

   Answer at the back of the book.

5. Two coins are tossed. What is the probability of one head and one tail?

   A 1/3

   B 1/2

   C 1/4

   Answer at the back of the book.

6. In the table of two dice, a total of 7 fills 6 cells and a total of 12 fills 1 cell. How many times as likely is a total of 7?

   A 12 times as likely

   B 5 times as likely

   C 6 times as likely

   Answer at the back of the book.

7. Two dice are rolled and added. What is the probability that the total is 9?

   Answer at the back of the book.

8. A coin is tossed and a dice is rolled. What is the probability of a tail and a number greater than 4?

   Answer at the back of the book.

KEEP IT

Copy this into your exercise book.

List every outcome in a table. Check that they are equally likely. P(event) = outcomes in the event ÷ outcomes in all. Two dice make 36 pairs, but the totals are not equally likely.

Draw: A 6 by 6 table of two dice with the cells that give a total of 7 hatched.

M4.6.2 CHAPTER 2

Relative frequency and expected results

A bottle top is dropped on a table. How can I find the chance that it lands top up?

YOUR GUESS

Pick one before you read on. It is fine to be wrong.

- ☐ **A** It can land two ways, so the chance is 1/2.
- ☐ **B** Drop it many times and see what fraction of the drops land top up.
- ☐ **C** Chance cannot be measured, so there is no way to find out.

IF THIS FEELS HARD, READ THIS FIRST

- Level 3, Statistics and chance, M3.7.8. If tossing a coin many times and comparing with what you expected looks new, read this first.
- Level 3, Statistics and chance, M3.7.7. If a probability as a fraction of equally likely outcomes looks new, read this first.

BY THE END YOU CAN

- Estimate a probability from the results of an experiment
- Find how many times to expect an outcome in 200 tries

Some events have no list of equally likely outcomes. A bottle top lands top up or the other way up, but the two are not equally likely. The only way to learn the chance is to try it.

Each go is a **trial**. The **relative frequency** of an event is the number of times it happened divided by the number of trials. It is an estimate of the probability.

DRAW IT FIRST

From a tally to an estimate

drops	top up	estimate
50	18	

1 Drop the bottle top 50 times. Count the drops that land top up. Here it is 18.

drops	top up	estimate
50	18	18 ÷ 50

2 Divide the number of times it happened by the number of trials: 18 ÷ 50.

drops	top up	estimate
50	18	0.36

3 The result, 0.36, is the relative frequency. It is your estimate of the probability.

WORKED EXAMPLE

A bottle top is dropped 200 times and lands top up 73 times. Estimate the probability that it lands top up.

1. Relative frequency = number of times it happened ÷ number of trials.
2. It happened 73 times in 200 trials.
3. 73 ÷ 200 = 0.365.
4. The estimate of the probability is 0.365.

Answer: 0.365

One experiment gives one estimate. Here are the same bottle top's results after more and more drops. The estimates move around, then settle near 0.37.

drops	top up	estimate
10	4	0.4
20	9	0.45
50	18	0.36
100	37	0.37
200	73	0.365
500	184	0.368

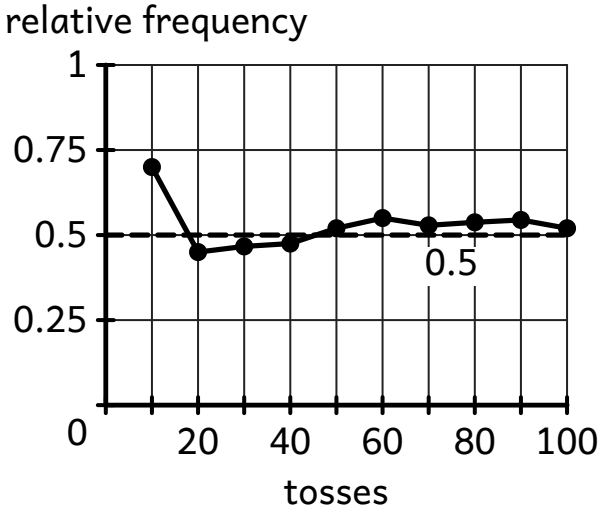
The estimate is the last column. Early estimates swing. Later ones settle.

A fair coin shows the same thing, and here we know the true probability is 0.5. This record holds the first 40 of 100 tosses. H is a head and T is a tail.

H H H T H H T T H H
T T T T H T T T H T
H H H T H T T T T H
H H H T T T H T T H

The first 40 tosses, in order. Heads came up 19 times.

Now work out the relative frequency of heads after 10 tosses, 20 tosses, and so on to 100. Plot each one against the number of tosses.



With few tosses the estimate swings. With more tosses it stays close to 0.5.

More trials usually give a better estimate. But a relative frequency never proves the probability. After 100 tosses the coin shows 0.52, not 0.5, and a new set of 100 tosses would give a different number.

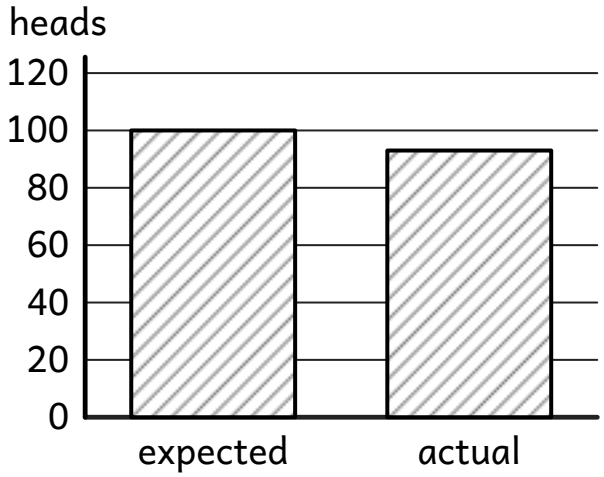
Now turn the idea round. If you know the probability, you can say what to expect. The **expected number** of times an event happens in n trials is the probability times n.

WORKED EXAMPLE

A spinner lands on blue with probability 3/10. It is spun 200 times. How many times should you expect blue?

1. Expected number = probability × number of trials.
2. Here that is 3/10 × 200.
3. 200 ÷ 10 = 20.
4. 3 × 20 = 60.
5. You expect blue about 60 times.

Answer: 60



A fair coin tossed 200 times: 100 heads expected, 93 found. That is normal.

WORKED EXAMPLE

A fair dice is rolled 200 times. How many sixes should you expect?

1. The probability of a six is 1/6.
2. Expected number = 1/6 × 200 = 200 ÷ 6. To 3 decimal places, 200 ÷ 6 = 33.333.
3. To 1 decimal place, 200 ÷ 6 = 33.3.
4. To the nearest whole number, you expect 33 sixes.
5. The expected number need not be a whole number, because it is an average over many sets of 200 rolls.

Answer: 33.3

TRY IT

Toss a coin and watch the estimate settle

You need: a coin, or a flat stone with a mark on one side, a pencil and paper.

1. Toss the coin 100 times. Write H or T each time.
2. After every 10 tosses, work out the relative frequency of heads so far.
3. Plot each value on a graph with tosses across and relative frequency up.
4. Compare your points at 10, 50 and 100 tosses. Which are closest to 0.5?

What you will see: The early points jump about. The later points move less and stay near 0.5. Nobody gets exactly 0.5 every time.

If you cannot try it, this is what happens.

tosses	heads	estimate
10	7	0.7
50	26	0.52
100	52	0.52

A WRONG IDEA MANY PEOPLE HAVE

~~The expected number is how many will happen.~~

Why it feels right: The word "expect" sounds like a promise, and 200 × 1/2 gives one clear number, 100.

The test that breaks it: Toss a fair coin 200 times, then do it again. The heads will be near 100 each time, but not exactly 100 each time.

What is true: The expected number is an average. The real number is usually close to it and often a little different.

KEY WORDS

trial
one go of an experiment, such as one toss of a coin.

relative frequency
the number of times an event happened divided by the number of trials. It estimates the probability.

expected number
the probability times the number of trials. It is an average, not a promise.

BACK TO YOUR GUESS

The answer is B: Drop it many times and see what fraction of the drops land top up.

The two ways are not equally likely, because the top and the bottom are different shapes. Counting outcomes does not work. Instead, drop it many times. The fraction of drops that land top up is an estimate of the probability.

CHECK YOURSELF

Try each one before you look. A wrong answer tells you where to go next.

1. A bottle top lands top up 30 times in 100 drops. What is the relative frequency of landing top up?

   A 0.7

   B 30

   C 0.3

   Answer at the back of the book.

2. You want a good estimate of the chance that a bottle top lands top up. Which plan is best?

   A Drop it as many times as you can, and divide.

   B Drop it a few times, then trust your feeling.

   C Drop it until the estimate looks right, then stop.

   Answer at the back of the book.

3. A spinner lands on green with probability 1/4. It is spun 200 times. How many times should you expect green?

   A 4

   B 150

   C 800

   D 50

   Answer at the back of the book.

4. A coin is tossed 200 times and lands heads 106 times. What does this tell you?

   A The coin must be unfair, because the heads are not exactly half.

   B It is close to the 100 heads that a fair coin is expected to give, so a small difference is normal.

   C The next 12 tosses must all be tails to even it out.

   Answer at the back of the book.

5. Which statement about relative frequency is true?

   A After 100 trials it always equals the probability.

   B It is exactly 0.5 for any fair coin.

   C It is an estimate that usually gets closer to the probability as the trials increase.

   Answer at the back of the book.

6. **From an earlier chapter**

   Two dice are rolled and added. What is the probability that the total is 4?

   Answer at the back of the book.

7. A bottle top lands top up 45 times in 250 drops. Estimate the probability that it lands top up.

   Answer at the back of the book.

8. A spinner lands on red with probability 3/5. It is spun 150 times. How many times should you expect red?

   Answer at the back of the book.

KEEP IT

Copy this into your exercise book.

Relative frequency = times it happened ÷ trials. It estimates a probability, and more trials usually improve it. Expected number = probability × trials, an average and not a promise.

Draw: A graph of relative frequency against trials, swinging at first and then settling near a dashed line.

M4.6.3 CHAPTER 3

The OR rule and the AND rule

> **A coin lands heads with chance 1/2 and a dice lands on 6 with chance 1/6. What is the chance of both?**

YOUR GUESS

Pick one before you read on. It is fine to be wrong.

- ☐ **A** Add them: 1/2 + 1/6 = 2/3.
- ☐ **B** Take away: 1/2 − 1/6 = 1/3.
- ☐ **C** Multiply them: 1/2 × 1/6 = 1/12.

IF THIS FEELS HARD, READ THIS FIRST

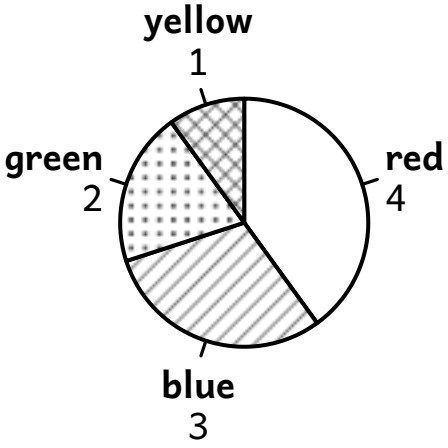
- Level 3, Statistics and chance, M3.7.7. If finding a probability from equally likely outcomes looks new, read this first.
- Level 3, Fractions, decimals and percentages, M3.3.3. If multiplying one fraction by another looks new, read this first.

BY THE END YOU CAN

- Add probabilities for events that cannot both happen
- Multiply probabilities for independent events

You often know the chance of each event. Then you need the chance of one event or another, or of both together. There are two rules, and each works only under a condition.

Take a spinner whose circle is split into tenths. Red takes 4 tenths, blue 3, green 2 and yellow 1. So P(red) = $^{4}/_{10}$ and P(blue) = $^{3}/_{10}$.

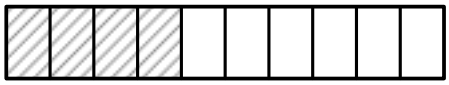


The spinner. One spin lands on one colour only.

What is the chance of red **or** blue? One spin cannot land on both. Events that cannot both happen are called **mutually exclusive**. For them, the chances simply add up.

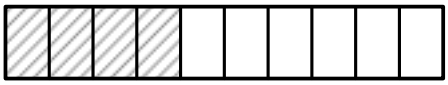
DRAW IT FIRST

Add the chances of red and blue

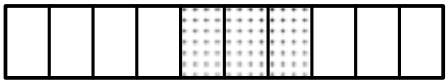


red 4/10

1 Shade the red part of the spinner: 4 of the 10 tenths.

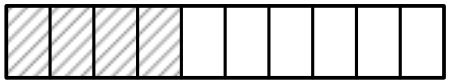


red 4/10

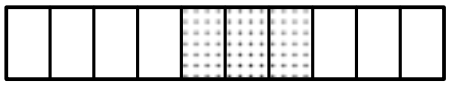


blue 3/10

2 Shade the blue part next to it: 3 more tenths. They do not overlap.



red 4/10



blue 3/10



red or blue 7/10

3 Red or blue is all the shaded tenths together: 4 + 3 = 7 out of 10.

This is the **OR rule** for events that cannot both happen: P(A or B) = P(A) + P(B).

WORKED EXAMPLE

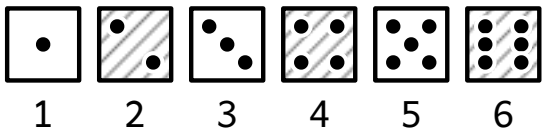
On the spinner, P(red) = 4/10 and P(blue) = 3/10. What is the probability of red or blue?

1. One spin cannot land on red and blue together, so the events cannot both happen.
2. P(red or blue) = P(red) + P(blue).
3. 4/10 + 3/10 = 7/10.

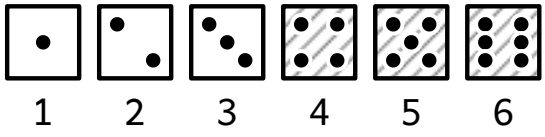
Answer: 7/10

The condition matters. Roll a dice. P(even) = $^3/_6$ and P(more than 3) = $^3/_6$. Adding gives 1, a certainty, but the dice can show a 1 or a 3, which are neither. What went wrong?

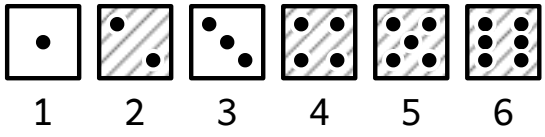
A score of 4 is even and more than 3. So is a score of 6. Adding counted those two scores twice. The events can both happen, so you may not simply add.



even



more than 3



even or more than 3

Even or more than 3 is 4 faces out of 6, not 6 out of 6.

A WRONG IDEA MANY PEOPLE HAVE

~~**To find the chance of A or B, always add the two chances.**~~

Why it feels right: The spinner example worked, and "or" sounds like "plus".

The test that breaks it: Roll a dice. Even and more than 3 each have chance 3/6. Their sum is 1, but a roll of 1 is neither. The faces 4 and 6 are in both, and adding counts them twice.

What is true: Add only for events that cannot both happen. If they can, take away the overlap. Chapter 6 shows how.

Now the other rule. A coin is tossed and a dice is rolled. The coin does not change the dice. Two events like this are **independent**: one happening does not change the chance of the other.

Half of all the cases have a head. Of those, 1 in 6 also has a six. So the chance of both is 1/6 of 1/2, which is 1/6 × 1/2 = 1/12. The table agrees.

	dice					
coin	**1**	**2**	**3**	**4**	**5**	**6**
H	H1	H2	H3	H4	H5	H6
T	T1	T2	T3	T4	T5	T6

One cell in 12 is a head and a 6. That is 1/2 × 1/6.

This is the **AND rule** for independent events: P(A and B) = P(A) × P(B).

WORKED EXAMPLE

A spinner lands on red with probability 2/5. A coin lands heads with probability 1/2. Both are done once. What is the probability of red and a head?

1. The spinner does not change the coin, so the events are independent.
2. P(red and head) = P(red) × P(head).
3. 2/5 × 1/2 = 2/10.
4. 2/10 = 1/5.

Answer: 1/5

WORKED EXAMPLE

Mei's bus is late with probability 1/10. Kofi's bus, in another town, is late with probability 1/5. What is the probability that both are late?

1. The two buses are in different towns, so one being late does not change the other.
2. The events are independent, so multiply.
3. Multiply the tops: 1 × 1 = 1.
4. Multiply the bottoms: 10 × 5 = 50.
5. So 1/10 × 1/5 = 1/50.

Answer: 1/50

TRY IT

Test the AND rule

You need: a coin, a dice, a pencil and paper.

1. Toss the coin and roll the dice together. Write down whether you got a head and a 6.
2. Do this 60 times. Count the "head and 6" results.
3. Divide that count by 60 for the relative frequency.
4. Compare it with 1/2 × 1/6, which is about 0.083.

What you will see: You should get about 5 results in 60. Yours may be 3 or 8. The rule gives the chance, and the experiment only estimates it.

If you cannot try it, this is what happens.

trials	both	estimate
60	4	0.067

A WRONG IDEA MANY PEOPLE HAVE

~~**After a coin lands heads five times, tails is more likely next.**~~

Why it feels right: It seems that the coin must "even things out".

The test that breaks it: Each toss is independent. The coin has no memory of the heads before. The chance of a head on the next toss is still 1/2.

What is true: Independent events do not change each other. Past tosses do not change the next toss.

KEY WORDS

mutually exclusive
events that cannot both happen at once, such as red and blue on one spin.

independent
events where one happening does not change the chance of the other.

OR rule

P(A or B) = P(A) + P(B), for events that cannot both happen.

AND rule

P(A and B) = P(A) × P(B), for independent events.

BACK TO YOUR GUESS

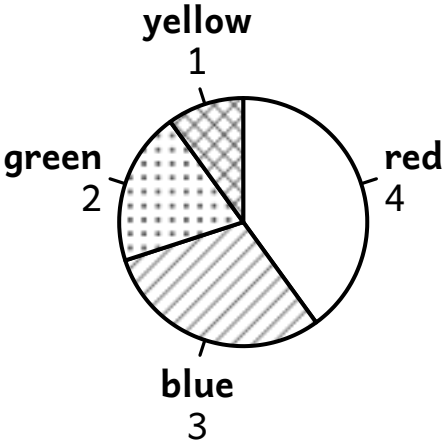
The answer is C: Multiply them: 1/2 × 1/6 = 1/12.

For "this and that", with events that do not affect each other, multiply. The chance of both is 1/2 × 1/6 = 1/12. You can check it on the table from chapter 1: one cell, H6, out of 12.

CHECK YOURSELF

Try each one before you look. A wrong answer tells you where to go next.

1. One spin lands on only one colour. What is the probability of green or yellow?



A 3/10

B 2/100

C 1/10

Answer at the back of the book.

2. A dice is rolled. For which two events can you add the chances to find the chance of one or the other?

   A An even score, or a score of more than three.

   B A score of more than two, or a score of less than five.

   C A prime score, or an odd score.

   D A score of one, or a score of six.

   Answer at the back of the book.

3. The chance that Priya's bus is late is 1/5. The chance that Tariq's bus is late is 1/4. The buses run on different roads. What is the probability that both are late?

   A 9/20

   B 1/20

   C 1/4

   Answer at the back of the book.

4. A fair coin has landed heads 5 times in a row. What is the chance that the next toss is a head?

   A Less than 1/2, because tails is due.

   B More than 1/2, because the coin is on a streak.

   C 1/2, because each toss is independent of the ones before.

   Answer at the back of the book.

5. Which statement about independent events is true?

   A One happening does not change the chance of the other.

   B They can never happen together.

   C They must have the same probability.

   Answer at the back of the book.

6. **From an earlier chapter**

   A spinner lands on red with probability 2/5. It is spun 300 times. How many times should you expect red?

   Answer at the back of the book.

7. Two events are independent. P(A) = 2/3 and P(B) = 3/5. What is P(A and B)?

   Answer at the back of the book.

8. On a spinner, the probability of red is 3/10 and of blue is 2/10. One spin cannot land on both. What is the probability of red or blue?

   Answer at the back of the book.

KEEP IT

Copy this into your exercise book.

A or B: add, but only if they cannot both happen. A and B: multiply, but only if they are independent. A coin has no memory.

Draw: A bar of ten tenths with the red part and the blue part shaded and added together.

M4.6.4 CHAPTER 4

Tree diagrams

> **A bag has 3 red and 2 blue counters. I take one, look, put it back, and take another. What is the chance of one of each colour?**

YOUR GUESS

Pick one before you read on. It is fine to be wrong.

- ☐ **A** 6/25, from 3/5 × 2/5.
- ☐ **B** 12/25, because red then blue and blue then red both count.
- ☐ **C** 1/2, because there are two colours.

IF THIS FEELS HARD, READ THIS FIRST

- Level 3, Statistics and chance, M3.7.7. If finding a chance from equally likely outcomes looks new, read this first.
- Level 4, Probability, M4.6.3. If adding chances for "or" and multiplying for "and" looks new, read this first.

BY THE END YOU CAN

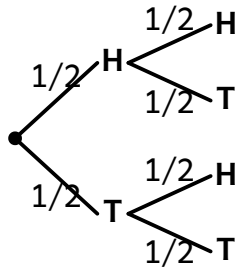
- Draw a tree diagram for two events
- Find probabilities by multiplying along the branches

A table lists the outcomes of two events well. When there are more events, or the chances are not equal, a **tree diagram** is better. It follows the events in order, one stage at a time.

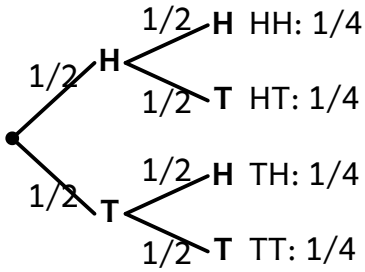
Start with a coin tossed twice. The first toss is H or T, and each has chance 1/2. From each of those, the second toss is H or T again.

DRAW IT FIRST

Draw *a* tree for two tosses of *a* coin



1 Draw the first toss: two branches, H and T, each marked 1/2. Then draw the second toss from the end of each.



2 At each end, write the outcome and its chance. Multiply the branches on the way: 1/2 × 1/2 = 1/4.

Two rules make a tree work. The branches that leave one point add up to 1, because one of them must happen. And along a path, you **multiply**, because each path is "this and then that".

WORKED EXAMPLE

A fair coin is tossed twice. What is the probability of two heads?

1. The path for two heads is H then H.
2. Each branch has probability 1/2.
3. Multiply along the path: 1/2 × 1/2 = 1/4.

Answer: 1/4

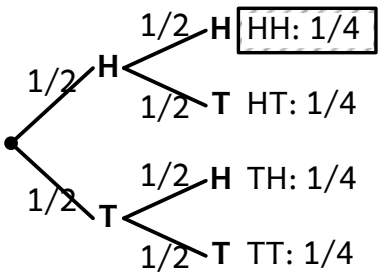
SEE IT THREE WAYS

With things

Toss a fair coin twice and write H or T for each toss. Do it a few times. Two heads comes up in 1 of every 4 goes, about.

As a picture

toss 1 \ toss 2	H	T
H	HH	HT
T	TH	TT



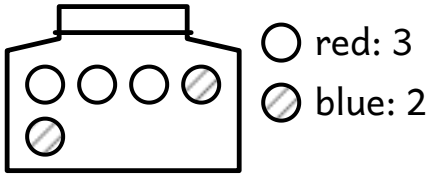
In words

List: the outcomes are HH, HT, TH and TT, and one of the 4 is HH. Table: one cell of 4 is HH. Tree: the path H then H has probability 1/2 times 1/2. All three ways give 1/4.

In symbols

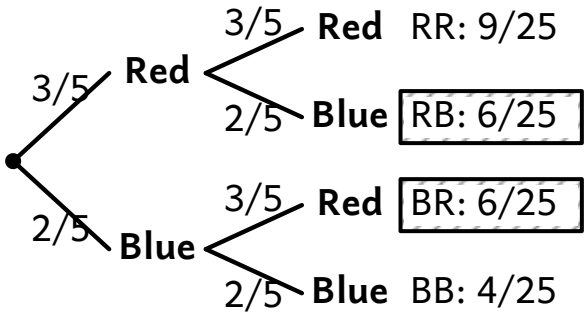
P(HH) = 1/4 = 1/2 × 1/2.

Now the bag. It holds 3 red and 2 blue counters, so P(red) = $^3/_5$ and P(blue) = $^2/_5$. We put the counter back each time, so the second pick has the same chances as the first.



the bag

The bag holds 5 counters. Each counter is put back after it is taken.



The two hatched ends are one of each colour. Their chances are 6/25 each.

To find the chance of an event, find every end where it happens. Work out each end by multiplying. Then add those ends, because the ends cannot happen together.

WORKED EXAMPLE

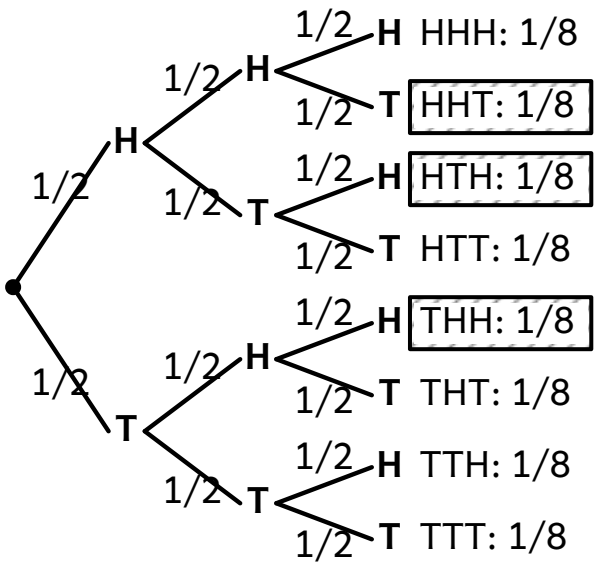
A counter is taken from the bag of 3 red and 2 blue, then put back, and a second is taken. What is the probability of one of each colour?

1. Red then blue: 3/5 × 2/5 = 6/25.
2. Blue then red: 2/5 × 3/5 = 6/25.
3. These are two different paths, so add them: 6/25 + 6/25 = 12/25.

Answer: 12/25

A good check: all the ends together must add to 1, because something must happen. Here 9/25 + 6/25 + 6/25 + 4/25 = 25/25 = 1.

A tree can have more stages. Toss a coin three times. Each stage doubles the number of ends, so there are 8 ends. Each end has chance 1/2 × 1/2 × 1/2 = 1/8.



Three tosses. The three hatched ends have exactly two heads.

WORKED EXAMPLE

A fair coin is tossed three times. What is the probability of exactly two heads?

1. The paths with exactly two heads are HHT, HTH and THH. That is 3 paths.
2. Each path has probability 1/2 × 1/2 × 1/2 = 1/8.
3. Add the three paths: 3 × 1/8 = 3/8.

Answer: 3/8

TRY IT

Test the tree with *a bag*

You need: 3 white stones *and* 2 dark stones, *a* cloth bag or *a* cup, *a* pencil *and* paper.

1. Put the stones in the bag. Pick one without looking, write its colour, *and* put it back.
2. Pick *again*, write the colour, *and* put it back.
3. Do this 50 times. Count the times you got one white *and* one dark.
4. The tree says 12/25 of the pairs, so you expect *about* 24 of 50.

What you will see: Your count is near 24, but not exactly 24. The tree gives the chance. The experiment only estimates it.

If you cannot try it, this is what happens.

pairs	one each	estimate
50	22	0.44

A WRONG IDEA MANY PEOPLE HAVE

~~**To find the chance of *a* path, *add* the branch chances.**~~

Why it feels right: We *add* chances for "or", so it seems we should *add* them here too.

The test that breaks it: Try red then red with chances 3/5 *and* 3/5. Adding gives 6/5, which is more than 1. No chance *can* be more than 1.

What is true: A path is "this *and* then that". Multiply *along* it. Add only where different paths end in the *same* event.

KEY WORDS

tree diagram
a diagram of branches that shows the outcomes of events in order, with *a* chance on *each* branch.

branch
one line of a tree diagram, marked with a probability.

BACK TO YOUR GUESS

The answer is B: 12/25, because red then blue and blue then red both count.

One of each can happen two ways: red then blue, and blue then red. Each way has chance 3/5 × 2/5 = 6/25. Add the two ways and you get 12/25. A tree diagram shows both ways at once.

CHECK YOURSELF

Try each one before you look. A wrong answer tells you where to go next.

1. Which labels can leave one point of a tree diagram?

   A 1/2 and 1/3

   B 1/4, 1/4 and 1/4

   C 3/5 and 3/5

   D 1/2, 1/3 and 1/6

   Answer at the back of the book.

2. A bag has 2 red and 3 blue counters. One is taken, put back, and a second is taken. What is the probability that both are red?

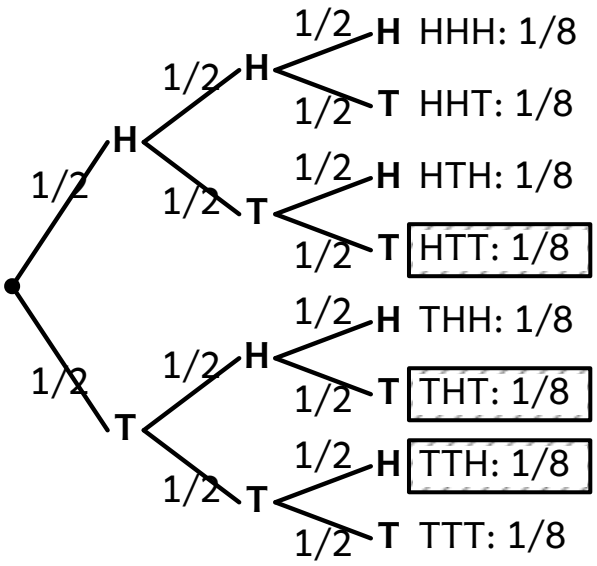
   A 2/5

   B 4/25

   C 4/5

   Answer at the back of the book.

3. A fair coin is tossed three times. What is the probability of exactly one head?



A 1/8

B 1/2

C 3/8

Answer at the back of the book.

4. On a tree, the branches to red then red are 3/5 and 3/5. Which is the chance of that path?

A Multiply the branches: 9/25.

B Add the branches: 6/5.

C Use one branch: 3/5.

Answer at the back of the book.

5. The chance of winning one game is 2/3. Two games are played, and they do not affect each other. What is the chance of exactly one win?

A 2/9

B 2/3

C 1/3

D 4/9

Answer at the back of the book.

6. **From an earlier chapter**

   Two events are independent. P(A) = 1/4 and P(B) = 1/3. What is P(A and B)?

   Answer at the back of the book.

7. A bag has 3 red and 1 blue counter. One is taken, put back, and a second is taken. What is the probability that both are red?

   Answer at the back of the book.

8. The chance of winning a game is 3/4. Two games are played, and they do not affect each other. What is the probability of exactly one win?

   Answer at the back of the book.

KEEP IT

Copy this into your exercise book.

A tree shows events in order. Branches from one point add to 1. Multiply along a path. Add the ends that make the event. All ends add to 1.

Draw: A tree for two tosses of a coin, with 1/2 on each branch and 1/4 at each end.

M4.6.5 CHAPTER 5

Picking without putting back

> **A bag has 5 red and 3 blue counters. I take one, keep it, and take a second. What is the chance that both are red?**

YOUR GUESS

Pick one before you read on. It is fine to be wrong.

- ☐ **A** 25/64, because 5/8 × 5/8.
- ☐ **B** 5/14, because 5/8 × 4/7.
- ☐ **C** 5/16, because 5/8 × 4/8.

IF THIS FEELS HARD, READ THIS FIRST

- Level 4, Probability, M4.6.4. If drawing a tree and multiplying along its branches looks new, read this first.
- Level 3, Statistics and chance, M3.7.7. If finding a chance from a count of equally likely outcomes looks new, read this first.

BY THE END YOU CAN

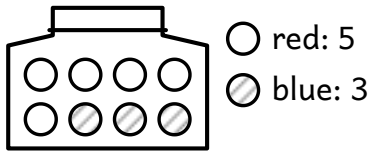
- Draw a tree diagram where the first pick changes the second
- Find the chance that two picks from a bag are both red

In chapter 4 each counter was put back, so the second pick had the same chances as the first. Now the counter is kept. This is picking **without putting back**.

The bag has changed, so the second pick has different chances. The two picks are **dependent**: the first changes the chance of the second.

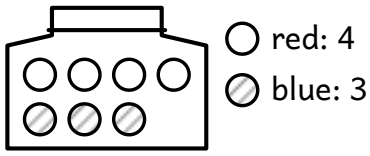
DRAW IT FIRST

How the bag changes



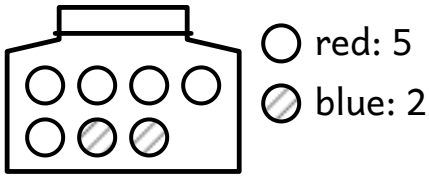
the start

1 The bag at the start has 5 red and 3 blue counters, 8 in all.



after a red is kept

2 A red counter is taken and kept. Now the bag has 4 red and 3 blue counters, 7 in all.



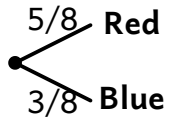
after a blue is kept

If a blue counter is taken instead, the bag has 5 red and 2 blue.

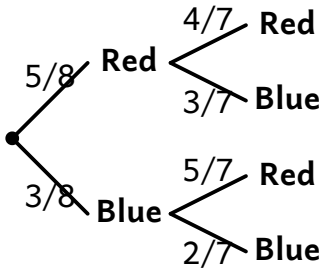
Read the chances for the second pick from the bag as it now is. After a red is kept: red is $^4/_7$ and blue is $^3/_7$. After a blue is kept: red is $^5/_7$ and blue is $^2/_7$. In each case the two add to 1.

DRAW IT FIRST

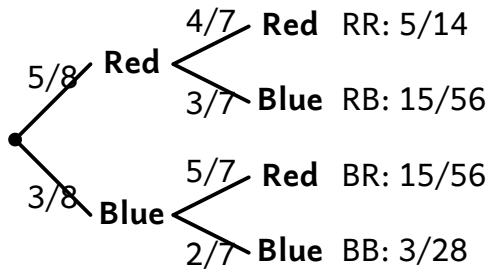
Draw the tree for picking without putting back



1 Draw the first pick: red 5/8 and blue 3/8.



2 Draw the second pick from each end. Use the bag as it is now: after red, 4/7 and 3/7. After blue, 5/7 and 2/7.



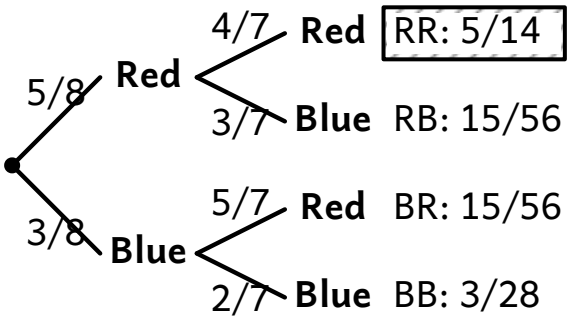
3 Multiply along each path to write the chance at each end.

WORKED EXAMPLE

A bag has 5 red and 3 blue counters. Two counters are taken without putting back. What is the probability that both are red?

1. The first counter is red with probability 5/8.
2. Then 4 red and 3 blue are left, so the second is red with probability 4/7.
3. P(both red) = 5/8 × 4/7 = 20/56.
4. 20/56 = 5/14.

Answer: 5/14



The tree for picking without putting back. The hatched end is both red.

WORKED EXAMPLE

The same bag, two counters taken and kept. What is the probability of one counter of each colour?

1. Red then blue: 5/8 × 3/7 = 15/56.
2. Blue then red: 3/8 × 5/7 = 15/56.
3. The two paths are different, so add: 15/56 + 15/56 = 30/56.
4. 30/56 = 15/28.

Answer: 15/28

Check the tree: the four ends must add to 1. Over 56, they are 20 + 15 + 15 + 6 = 56, so 56/56 = 1. This is a good way to catch a slip.

Compare with putting the counter back. The chance of both red falls from 25/64 to 5/14, because taking a red leaves fewer reds in the bag. The chance of one of each rises from 15/32 to 15/28.

event	put back	kept
both red	25/64	5/14
one each	15/32	15/28
both blue	9/64	3/28

The same bag, put back or kept. Keeping the first counter makes "same colour" less likely.

TRY IT

Pick two stones and keep them

You need: 5 white stones and 3 dark stones, a cloth bag or a cup, a pencil and paper.

1. Put the 8 stones in the bag. Pick two without looking, and write the colours.
2. Put both back, shake the bag, and do it again.
3. Do this 40 times. Count the times both were white.
4. The tree says 5/14 of the pairs, so you expect about 14 of 40.

What you will see: Your count is near 14. If you put the first stone back each time, you would see more pairs of white stones, about 16 in 40.

If you cannot try it, this is what happens.

pairs	two white	estimate
40	15	0.375

A WRONG IDEA MANY PEOPLE HAVE

~~The second pick has the same chances as the first.~~

Why it feels right: The bag looks the same, and the same colours are in it.

The test that breaks it: Take a red counter and keep it. The bag now has 7 counters, not 8, and only 4 are red. The chance of red is 4/7, not 5/8.

What is true: When a counter is kept, change both numbers for the second pick. There is one fewer in all, and one fewer of the colour taken.

KEY WORDS

without putting back
taking a counter and keeping it out, so the next pick is from a smaller bag.

dependent
events where one happening changes the chance of the other.

BACK TO YOUR GUESS

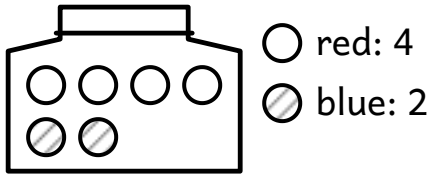
The answer is B: 5/14, because 5/8 × 4/7.

After a red counter is taken and kept, 7 counters are left, and only 4 of them are red. So the second pick has chance 4/7, not 5/8. The chance of both is 5/8 × 4/7 = 20/56 = 5/14.

CHECK YOURSELF

Try each one before you look. A wrong answer tells you where to go next.

1. A bag has 4 red and 2 blue counters. A red counter is taken and kept. What is the chance that the next counter is red?



the bag

   A 4/6

   B 3/5

   C 3/6

   Answer at the back of the book.

2. A bag has 4 red and 2 blue counters. Two are taken without putting back. What is the probability that both are red?

   A 4/9

   B 2/3

   C 2/5

   Answer at the back of the book.

3. A bag has 6 red and 3 blue counters. Two are taken without putting back. The first is blue. What is the chance that the second is red?

   A 3/4

   B 2/3

   C 5/8

   Answer at the back of the book.

4. A tree for two picks has three ends with chances 12/42, 12/42 and 12/42. The fourth is missing. What is it?

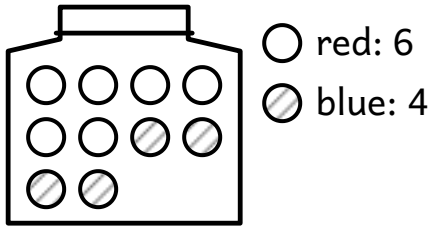
   A 9/42

   B 6/42

   C 4/42

   Answer at the back of the book.

5. A bag has 6 red and 4 blue counters. Two are taken without putting back. What is the probability that both are blue?



the bag

A 4/25

B 2/5

C 1/3

D 2/15

Answer at the back of the book.

6. **From an earlier chapter**

   A bag has 2 red and 1 blue counter. One is taken, put back, and a second is taken. What is the probability that both are red?

   Answer at the back of the book.

7. A bag has 4 red and 3 blue counters. Two are taken without putting back. What is the probability that both are red?

   Answer at the back of the book.

8. A bag has 3 red and 2 blue counters. Two are taken without putting back. What is the probability that the two are different colours?

   Answer at the back of the book.

KEEP IT

Copy this into your exercise book.

Without putting back, the bag shrinks. Second pick: one fewer counter in all, and one fewer of the colour taken. Multiply along paths. All ends add to 1.

Draw: A tree for two picks from a bag of 5 red and 3 blue, with 5/8, 3/8 and the changed second branches.

M4.6.6 CHAPTER 6

Venn diagrams and sets

> **In a class of 30, 17 like football, 12 like running, and 5 like both. What is the chance that a pupil chosen at random likes football or running?**

YOUR GUESS

Pick one before you read on. It is fine to be wrong.

- ☐ **A** 29/30, from 17/30 + 12/30.
- ☐ **B** 4/5, because 5 pupils are counted twice and must be taken off once.
- ☐ **C** 1/6, the pupils who like both.

IF THIS FEELS HARD, READ THIS FIRST

- Level 4, Probability, M4.6.3. If adding chances for "or" looks new, read this first.
- Level 3, Statistics and chance, M3.7.7. If finding a chance from a count of equally likely outcomes looks new, read this first.

BY THE END YOU CAN

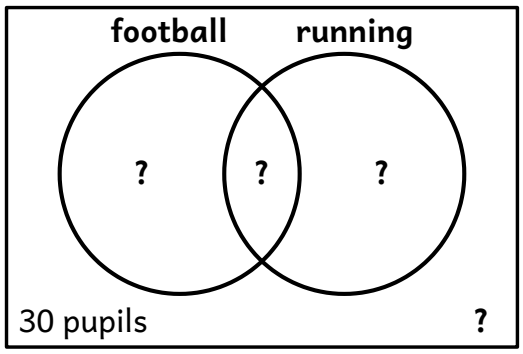
- Draw a Venn diagram for two sets
- Find P(A or B) and P(A and B) from a Venn diagram
- Use set notation like A ∪ B and A ∩ B

A **set** is a collection of things, such as the pupils who like football. A **Venn diagram** shows sets as circles inside a box. The box holds everything we are talking about, and each circle holds one set.

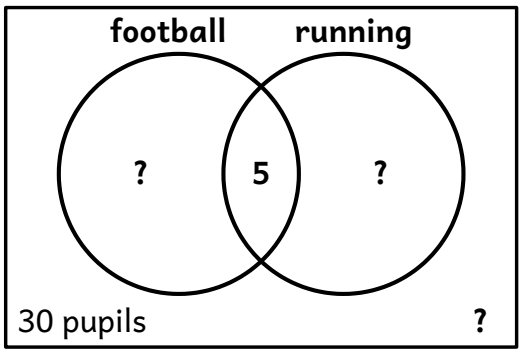
Let A be the set of pupils who like football and B the set who like running. Some pupils like both, so the circles overlap.

DRAW IT FIRST

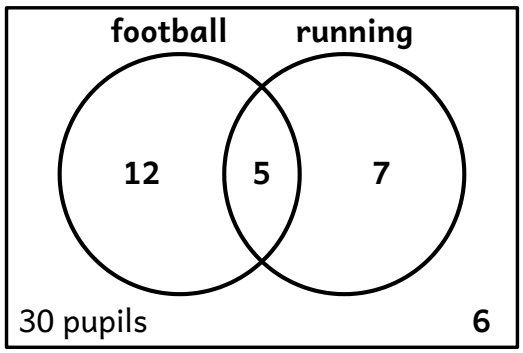
Fill in *a* Venn di*a*gram



1 Draw *a* box for *a*ll 30 pupils. Draw two overlapping circles, one for football *a*nd one for running.



2 Start in the overlap. 5 pupils like both, so write 5 there.



3 Football has 17 in *a*ll, so 17 − 5 = 12 *a*re in its circle only. Running has 12 in *a*ll, so 12 − 5 = 7 *a*re in its circle only. The rest, 30 − 24 = 6, *a*re outside both.

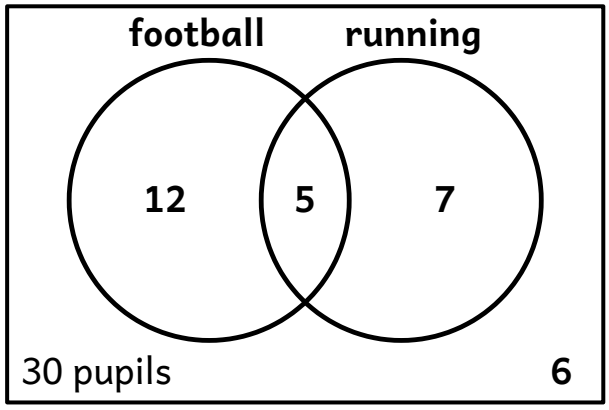
Always begin with the overlap. The totals for each set, 17 *a*nd 12, include the overlap, so each circle needs the overlap taken off.

WORKED EXAMPLE

In a class of 30, 17 like football, 12 like running, and 5 like both. How many pupils like neither?

1. Both: 5 pupils. Football only: 17 − 5 = 12 pupils.
2. Running only: 12 − 5 = 7 pupils.
3. Football or running or both: 12 + 5 + 7 = 24 pupils.
4. Neither: 30 − 24 = 6 pupils.

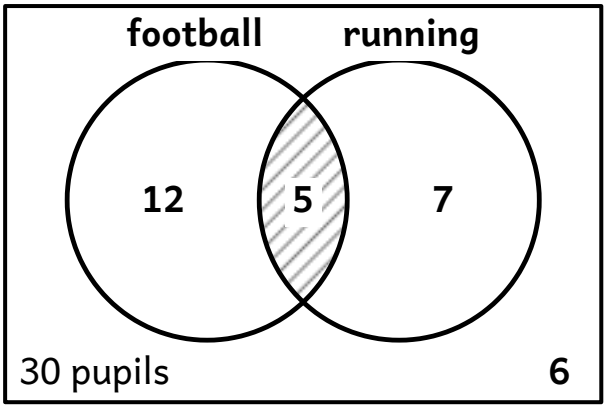
Answer: 6



The four regions add up to 30: 12 + 5 + 7 + 6.

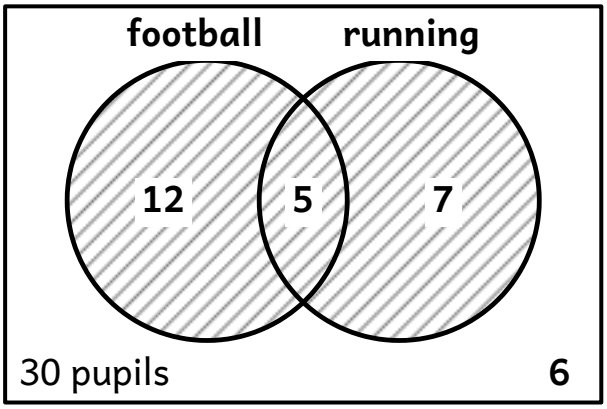
Now read chances from the diagram. Choose one pupil at random. Each of the 30 pupils is equally likely, so a chance is a count of pupils divided by 30.

The **intersection** of A and B is the pupils in both circles. It is written $A \cap B$. Its probability is $P(A \cap B) = {}^{5}/_{30} = {}^{1}/_{6}$. This is the chance of A and B.



A ∩ B: the overlap, in both circles. 5 of 30.

The **union** of A and B is the pupils in A or B or both. It is written A ∪ B. All three regions that touch a circle count, so P(A ∪ B) = $^{24}/_{30}$ = $^{4}/_{5}$. This is the chance of A or B.



A ∪ B: everything in either circle. 24 of 30.

WORKED EXAMPLE

A pupil is chosen at random from the class of 30. What is the probability that the pupil likes football or running?

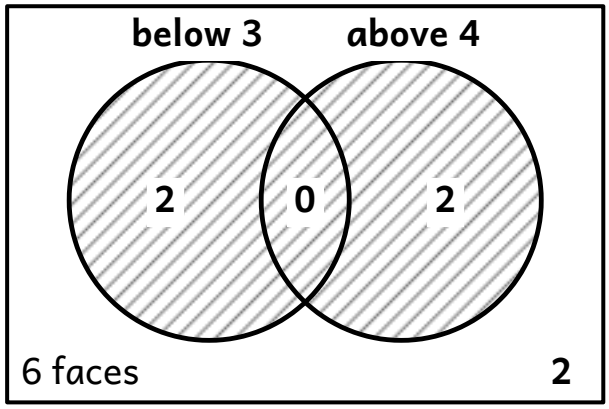
1. A pupil likes football or running when the pupil is in either circle.
2. Count the pupils in either circle: 12 + 5 + 7 = 24.
3. P(A ∪ B) = 24/30.
4. 24/30 = 4/5.

Answer: 4/5

You can also reach this from the totals. P(A) = $^{17}/_{30}$ and P(B) = $^{12}/_{30}$. Adding counts the overlap twice, so subtract it once.

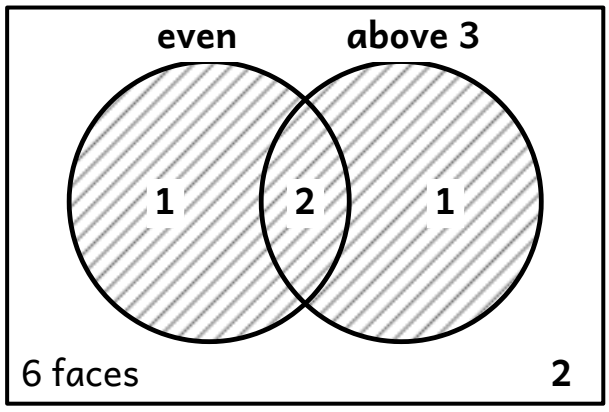
P(A ∪ B) = P(A) + P(B) − P(A ∩ B). Here that is 17/30 + 12/30 − 5/30 = 24/30. This matches the diagram.

In chapter 3 you added chances for events that cannot both happen. Then the overlap is empty, P(A ∩ B) = 0, and nothing is subtracted. The rule above covers that case too.



A dice score below 3 and a score above 4 cannot both happen. The overlap holds 0, so you add: 2/6 + 2/6.

Now the dice from chapter 3. Even scores are 2, 4 and 6. Scores above 3 are 4, 5 and 6. The scores 4 and 6 are in both.



Even or above 3 is 1 + 2 + 1 = 4 faces out of 6, which is 2/3.

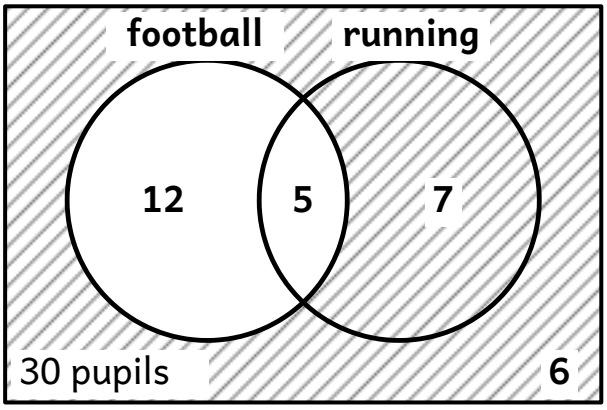
One more piece of notation. A′ means "not A": everything in the box outside circle A. Its chance is P(A′) = 1 − P(A).

WORKED EXAMPLE

A pupil is chosen at random from the class of 30. What is the probability that the pupil does not like football?

1. A′ is every pupil outside the football circle.
2. P(A) = 17/30, so A′ is the other 30 − 17 = 13 pupils.
3. P(A′) = 13/30.
4. Check with 1 − P(A) = 30/30 − 17/30 = 13/30.

Answer: 13/30



A′: everything outside the football circle. 13 of 30.

TRY IT

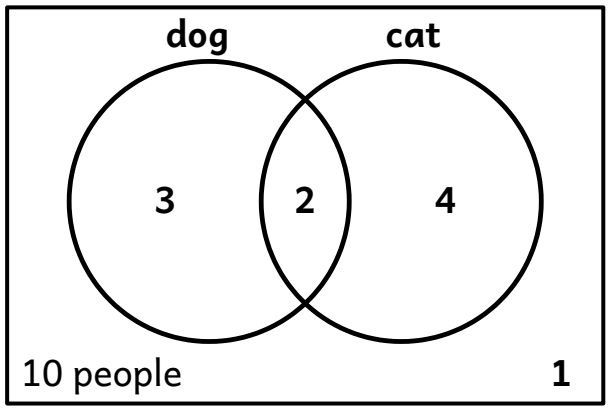
A Venn diagram of ten people

You need: a piece of paper, a pencil, ten people to ask, such as your family and friends.

1. Ask each person two questions: do you have a dog at home, and do you have a cat?
2. Draw a box and two overlapping circles, one for dogs and one for cats.
3. Write each person in the right region. Someone with both goes in the overlap.
4. Count each region and check that the counts add to 10.

What you will see: Every person goes in exactly one region. That is why the four counts must add up to the number of people.

If you cannot try it, this is what happens.



A WRONG IDEA MANY PEOPLE HAVE

~~Write 17 in the football circle and 12 in the running circle.~~

Why it feels right: The question gives 17 for football and 12 for running, and a circle is where football pupils go.

The test that breaks it: The regions would then hold 17 + 5 + 12 + 6 = 40 pupils, but the class has only 30. The 17 already includes the 5 in the overlap.

What is true: Start with the overlap. In the rest of each circle, write the total for that set minus the overlap.

KEY WORDS

set
a collection of things, such *as* the pupils who like football.

Venn diagram
a diagram of circles in *a* box that shows how sets overlap.

union
everything in A or B or both. It is written A ∪ B.

intersection
everything in both A and B. It is written A ∩ B.

BACK TO YOUR GUESS

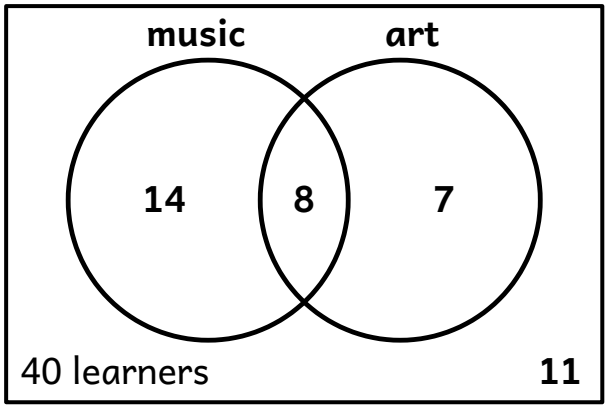
The answer is B: 4/5, because 5 pupils are counted twice and must be taken off once.

The 5 pupils who like both are in the 17 and in the 12, so adding counts them twice. Take 5 off once: 17 + 12 − 5 = 24 pupils like football or running. 24/30 = 4/5.

CHECK YOURSELF

Try each one before you look. A wrong answer tells you where to go next.

1. One learner is chosen at random from the 40. What is the probability that the learner is in music or art or both?



A 37/40

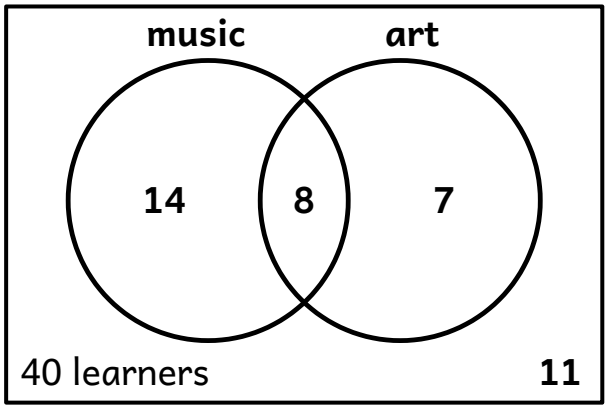
B 22/40

C 29/40

D 29/37

Answer at the back of the book.

2. One learner is chosen at random from the 40. What is the probability that the learner is in both music and art?



A 1/5

B 8/22

C 8/29

Answer at the back of the book.

3. In set notation, what does $A \cap B$ mean?

   A Everything in A or in B or in both.

   B Everything in both A and B.

   C Everything that is not in A.

   Answer at the back of the book.

4. 14 learners play chess, and 6 of them also play draughts. In a Venn diagram, what number goes in the region for chess only?

   A 14

   B 20

   C 6

   D 8

   Answer at the back of the book.

5. $P(A) = 7/10$. What is the probability of A', which means not A?

   A 7/10

   B 3/7

   C 3/10

   Answer at the back of the book.

6. **From an earlier chapter**

   A bag has 3 red and 2 blue counters. Two are taken without putting back. What is the probability that both are red?

   Answer at the back of the book.

7. In a group of 40 learners, 25 play football, 18 play netball and 9 play both. How many play neither?

   Answer at the back of the book.

8. $P(A) = 0.6$, $P(B) = 0.3$ and $P(A \cap B) = 0.1$. What is $P(A \cup B)$?

   Answer at the back of the book.

KEEP IT

Copy this into your exercise book.

Start with the overlap. Regions add to the whole. A ∩ B is both. A ∪ B is either or both. P(A ∪ B) = P(A) + P(B) − P(A ∩ B). P(A′) = 1 − P(A).

Draw: A box with two overlapping circles, the overlap filled in first, and the four counts adding to the whole.

M4.6.7 CHAPTER 7

Conditional probability

> **In a class of 60, some pupils walk to school and some take the bus. If I pick a girl, what is the chance that she walks?**

YOUR GUESS

Pick one before you read on. It is fine to be wrong.

- ☐ **A** 32/60, because 32 of the 60 pupils walk.
- ☐ **B** 18/60, because 18 girls walk.
- ☐ **C** 18/30, because I only look at the 30 girls.

IF THIS FEELS HARD, READ THIS FIRST

- Level 4, Probability, M4.6.6. If finding P(A and B) from a diagram looks new, read this first.
- Level 4, Probability, M4.6.5. If the second pick changing after the first looks new, read this first.
- Level 3, Fractions, decimals and percentages, M3.3.3. If dividing one fraction by another looks new, read this first.

BY THE END YOU CAN

- Find a probability given that something has already happened
- Use a two way table to find a conditional probability

Sometimes you already know something. You are told that a pupil is a girl, or that the first counter was red. This changes the chance of what comes next.

The chance of B when you know that A has already happened is called a **conditional probability**. We write it P(B given A).

A two way table is the best way to see it. Here are the 60 pupils, sorted by girl or boy and by walk or bus. The totals are worked out for you.

Pupils	Walk	Bus	Total
Girls	18	12	30
Boys	14	16	30
Total	32	28	60

The class of 60. 30 girls, 30 boys, 32 who walk and 28 who take the bus.

DRAW IT FIRST

Find P(walks given girl) from the table

Pupils	Walk	Bus	Total
Girls	18	12	30
Boys	14	16	30
Total	32	28	60

1 Start with the table of all 60 pupils.

Pupils	Walk	Bus	Total
Girls	18	12	30
Boys	14	16	30
Total	32	28	60

2 You know the pupil is a girl, so look only at the Girls row. It holds 30 pupils.

Pupils	Walk	Bus	Total
Girls	18	12	30
Boys	14	16	30
Total	32	28	60

3 In that row, count the girls who walk: 18. The chance is 18 out of 30.

WORKED EXAMPLE

A girl is chosen at random from the class of 60. What is the probability that she walks?

1. We know she is a girl, so look at the Girls row only. It has 18 + 12 = 30 pupils.
2. In that row, 18 pupils walk.
3. P(walks given girl) = 18/30.
4. 18/30 = 3/5.

Answer: 3/5

Notice what changed. The chance that a pupil in the whole class walks is $^{32}/_{60}$, which is $^{8}/_{15}$. For a girl it is $^{3}/_{5}$. Knowing the pupil is a girl changed the chance.

Why does the table method work? The chance that a pupil is a girl and walks is P(A and B) = $^{18}/_{60}$. The chance of a girl is P(A) = $^{30}/_{60}$. Divide the first by the second and the 60 cancels.

This gives the rule P(B given A) = P(A and B) ÷ P(A). Here, 18/60 ÷ 30/60 = 18/30. Dividing by P(A) shrinks the world to the pupils in A.

WORKED EXAMPLE

A pupil who walks is chosen at random. What is the probability that the pupil is a girl?

1. We know the pupil walks, so look at the Walk column only. It has 18 + 14 = 32 pupils.
2. In that column, 18 are girls.
3. P(girl given walks) = 18/32.
4. 18/32 = 9/16.

Answer: 9/16

Pupils	Walk	Bus	Total
Girls	18	12	30
Boys	14	16	30
Total	32	28	60

P(girl given walks): look only at the Walk column. 18 of its 32.

Compare the two answers. P(walks given girl) = 3/5. P(girl given walks) = 9/16. They are different, because they look at different groups: the girls, and the walkers.

A WRONG IDEA MANY PEOPLE HAVE

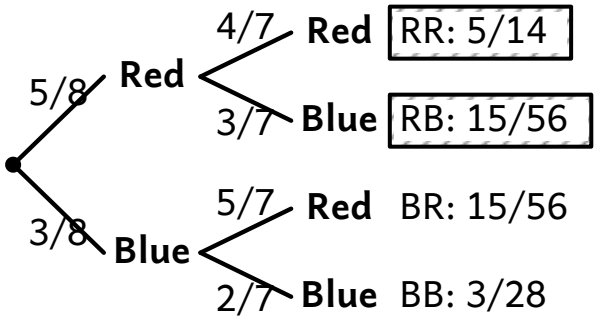
~~P(A given B) is the same as P(B given A).~~

Why it feels right: Both mention the same two events, and A and B seem to be sharing the same overlap.

The test that breaks it: The overlap is the same, 18 pupils, but it is divided by different totals. 18 out of 30 girls walk, while 18 out of 32 walkers are girls.

What is true: The event after "given" is the one you know. It sets the group you divide by. Always check which one it is.

You have met conditional chances before. In chapter 5, a bag held 5 red and 3 blue counters, and the first counter was kept. The branches for the second pick are conditional chances.



After a red first, the second branches are 4/7 and 3/7. They are P(second given first is red).

WORKED EXAMPLE

Use the rule P(B given A) = P(A and B) ÷ P(A). A is "the first counter is red" and B is "the second is red". What is P(B given A)?

1. P(A and B) = P(red, red) = 5/8 × 4/7 = 5/14.
2. P(A) = P(first is red) = 5/8.
3. P(B given A) = 5/14 ÷ 5/8 = 5/14 × 8/5 = 40/70.
4. 40/70 = 4/7. This is the branch on the tree.

Answer: 4/7

There is a link with independence. Events are independent when knowing one does not change the chance of the other: P(B given A) = P(B). In the pupil table, P(walks) = 8/15 but P(walks given girl) = 3/5. These differ, so being a girl and walking are not independent in this class.

TRY IT

Make a two way table of twenty people

You need: a pencil and paper, twenty people to ask.

1. Ask each person two questions: do you drink tea, and do you drink milk?
2. Draw a table with Tea and No tea down the side, and Milk and No milk across the top.
3. Make a mark in the right cell for each person, then count and add the totals.
4. Find P(milk given tea): look only at the tea row.

What you will see: The row for tea has its own total. That total, and not 20, is what you divide by.

If you cannot try it, this is what happens.

	Milk	No milk	Total
Tea	6	5	11
No tea	4	5	9
Total	10	10	20

KEY WORDS

conditional probability
the chance of one event when you already know that another has happened.

given
the word that names what you already know. P(B given A) is the chance of B when A has happened.

BACK TO YOUR GUESS

The answer is C: 18/30, because I only look at the 30 girls.

You already know the pupil is a girl, so only the 30 girls matter. Of those 30, 18 walk. The chance is 18/30 = 3/5.

CHECK YOURSELF

Try each one before you look. A wrong answer tells you where to go next.

1. One of the staff is chosen at random. What is the probability that this person has tea?

People	Tea	Juice	Total
Staff	18	6	24
Visitors	4	22	26
Total	22	28	50

   A 3/4

   B 9/25

   C 11/25

   Answer at the back of the book.

2. One of the people who have tea is chosen at random. What is the probability that this person is staff?

People	Tea	Juice	Total
Staff	18	6	24
Visitors	4	22	26
Total	22	28	50

   A 3/4

   B 9/11

   C 9/25

   Answer at the back of the book.

3. What does P(A given B) mean?

   A The chance of A and B together.

   B The chance of B first and then A.

   C The chance of A when you know that B has happened.

   Answer at the back of the book.

4. P(A and B) = 1/5 and P(A) = 2/5. What is P(B given A)?

   A 1/5

   B 2/5

   C 2/25

   D 1/2

   Answer at the back of the book.

5. In the staff table, P(tea given staff) is 3/4 and P(tea) is 11/25. What does this show?

   A Being staff and having tea are not independent, because knowing the person is staff changes the chance of tea.

   B They are independent, because both are chances of tea.

   C The table has a mistake, because the two chances should match.

   Answer at the back of the book.

6. **From an earlier chapter**

   In a group of 50, 30 play football, 25 play netball and 12 play both. How many play neither?

   Answer at the back of the book.

7. In a group of 60 learners, 24 are in the choir. 9 learners are in the choir and also play the drum. A learner in the choir is chosen. What is the probability that this learner plays the drum?

   Answer at the back of the book.

8. P(A and B) = 3/20 and P(A) = 6/20. What is P(B given A)?

   Answer at the back of the book.

KEEP IT

Copy this into your exercise book.

P(B given A): look only at the row or column of A. P(B given A) = P(A and B) ÷ P(A). P(B given A) and P(A given B) differ. Independent: P(B given A) = P(B).

Draw: A two way table with the row of the known event hatched and the matching cell marked.

M4.6.8 CHAPTER 8

Counting choices

> **Mei has 3 scarves and 2 hats. How many different outfits can she make with one scarf and one hat?**

YOUR GUESS

Pick one before you read on. It is fine to be wrong.

- ☐ **A** 3, one for each scarf.
- ☐ **B** 5, because 3 + 2.
- ☐ **C** 6, because each of the 3 scarves goes with each of the 2 hats.

IF THIS FEELS HARD, READ THIS FIRST

- Level 4, Probability, M4.6.1. If listing every outcome of two events looks new, read this first.
- Level 3, Long multiplication and division, M3.2.1. If multiplying whole numbers in your head looks new, read this first.

BY THE END YOU CAN

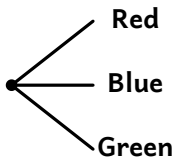
- Count outcomes with the product rule
- Find how many ways there are to arrange 4 different letters

To find a probability by counting, you often need to know how many outcomes there are in all. Listing them works for small cases. For bigger cases, you need a faster way to count.

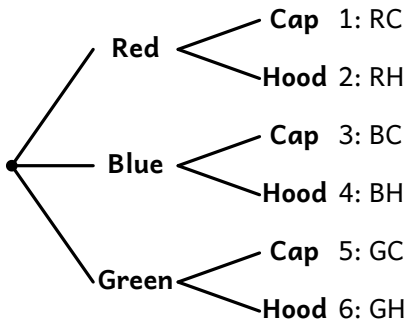
Start with Mei's clothes. A tree with no chances on its branches lists the outfits. Each scarf has two branches, one for each hat.

DRAW IT FIRST

List the outfits *as a* tree



1 Draw the first choice: three branches for the three scarves.



2 From each scarf, draw two branches for the two hats. Number the ends: there are 6 outfits.

Notice that the tree has 3 groups of 2 ends. The number of ends is 3 × 2. This is the **product rule**. Suppose one choice can be made in *a* ways, and another in b ways. Then both can be made in *a* × b ways.

scarf hat

3 × **2** = **6**

One box for each choice. Multiply the numbers in the boxes.

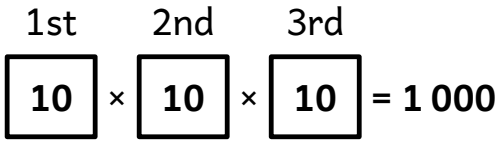
WORKED EXAMPLE

Mei has 3 scarves and 2 hats. How many outfits can she make with one scarf and one hat?

1. There are 3 ways to choose the scarf.
2. For each scarf there are 2 ways to choose the hat.
3. Multiply: 3 × 2 = 6 outfits.

Answer: 6

The rule works for any number of choices. Multiply the number of ways for each one. A lock has 3 wheels, and each wheel shows a digit from 0 to 9. Each wheel has 10 ways.



A lock with three wheels of ten digits. 10 × 10 × 10 codes.

WORKED EXAMPLE

A lock has 3 wheels. Each wheel shows a digit from 0 to 9. How many different codes are there?

1. Each wheel has 10 ways, and a wheel can repeat a digit shown by another wheel.
2. The first two wheels together have 10 × 10 = 100 ways.
3. With the third wheel, 100 × 10 = 1 000 codes.

Answer: 1 000

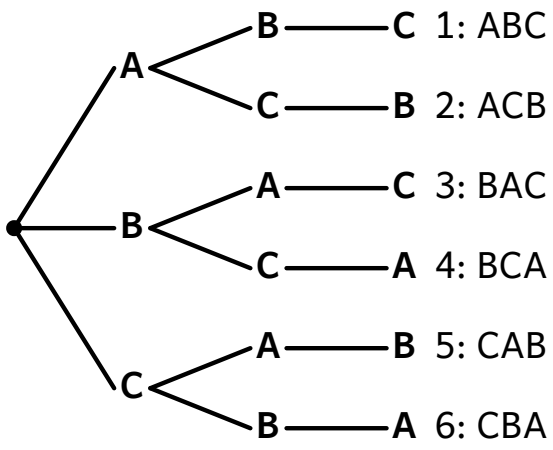
WORKED EXAMPLE

Joseph guesses a code for the lock at random. What is the probability that he opens it on his first try?

1. All 1 000 codes are equally likely.
2. Only one code opens the lock.
3. P(opens it) = $^{1}/_{1000}$.
4. As a decimal, 1 ÷ 1 000 = 0.001.

Answer: $^{1}/_{1000}$

Now a different kind of choice: putting things in order. Each way of doing it is an ***arrangement***. How many arrangements of the letters A, B and C in a row are there, each letter used once? A tree lists them. After you choose the first letter, only two letters are left, and after the second, only one.



Three letters in a row: 6 arrangements.

The number of choices falls each time, because a letter that is used cannot be used again. So the number of ways is 3 × 2 × 1 = 6, which agrees with the tree.

1st		2nd		3rd	
3	×	**2**	×	**1**	**= 6**

Three letters: 3 choices, then 2, then 1.

Four letters A, B, C and D work the same way. There are 4 ways to choose the first letter. Then 3 are left for the second, 2 for the third and 1 for the last.

1st		2nd		3rd		4th	
4	×	**3**	×	**2**	×	**1**	**= 24**

Four different letters can be arranged in 4 × 3 × 2 × 1 = 24 ways.

WORKED EXAMPLE

In how many ways can 4 different letters be arranged in a row?

1. The first place can be filled in 4 ways.
2. Then 3 letters are left. The first two places can be filled in 4 × 3 = 12 ways.
3. Then 2 letters are left, so the first three places can be filled in 12 × 2 = 24 ways.
4. The last place has 1 way, so 4 × 3 × 2 × 1 = 24.

Answer: 24

Counting gives probabilities. If the 4 letters are put in a random order, all 24 arrangements are equally likely. The chance that they come out as ABCD is 1 out of 24.

TRY IT

Find all 24 arrangements

You need: four small pieces of paper, a pencil.

1. Write A, B, C and D on the four pieces.
2. Put A first. List every way to arrange B, C and D after it. There should be 6.
3. Do the same with B first, then C first, then D first.
4. Add up. Each first letter gives 6, so the total is 4 × 6 = 24.

What you will see: Each first letter gives the same count, 3 × 2 × 1 = 6. That is the product rule at work.

If you cannot try it, this is what happens.

first	ways
A	6
B	6
C	6
D	6

A WRONG IDEA MANY PEOPLE HAVE

~~**To count the outfits, add the numbers of choices: 3 + 2.**~~

Why it feels right: We add when we put groups together, and here there are two groups, scarves and hats.

The test that breaks it: Add and you get 5, but the tree shows 6 outfits. Each scarf is worn with each hat, so every scarf is counted twice, once for each hat.

What is true: When you make one choice and then another, multiply. Add only when you choose from one group or from the other, but not both.

A WRONG IDEA MANY PEOPLE HAVE

~~**Each place in a row of 4 letters has 4 choices, so there are 4 × 4 × 4 × 4 ways.**~~

Why it feels right: The first place does have 4 choices, and it seems the others should too.

The test that breaks it: A letter used in the first place cannot be used again. The second place has only 3 letters left, then 2, then 1.

What is true: When nothing can be used twice, each choice has one fewer option than the last: 4 × 3 × 2 × 1.

KEY WORDS

product rule
if one choice has *a* ways and another has b ways, both together have *a* × b ways.

arrangement
a way of putting things in order, such as the letters ABCD in *a* row.

BACK TO YOUR GUESS

The answer is C: 6, because each of the 3 scarves goes with each of the 2 hats.

Each scarf can be worn with either hat. That makes 2 outfits for the first scarf, 2 for the second and 2 for the third. 2 + 2 + 2 = 6, which is 3 × 2.

CHECK YOURSELF

Try each one before you look. A wrong answer tells you where to go next.

1. A pupil chooses one of 4 books and one of 5 pens. How many different pairs can the pupil make?

   A 9

   B 16

   C 25

   D 20

   Answer at the back of the book.

2. A lock has 3 wheels. Each wheel shows one digit from 0 to 4. How many different codes are there?

   A 15

   B 60

   C 125

   Answer at the back of the book.

3. In how many ways can 3 different people stand in a line, one person at each place?

   A 3

   B 6

   C 27

   Answer at the back of the book.

4. Five runners finish a race. In how many ways can first, second and third places be filled?

   A 60

   B 15

   C 125

   Answer at the back of the book.

5. The letters A, B, C and D are put in a random order. What is the probability that the order starts with AB?

   A 1/24

   B 1/12

   C 1/16

   D 1/4

   Answer at the back of the book.

6. **From an earlier chapter**

   P(A and B) = 4/20 and P(A) = 10/20. What is P(B given A)?

   Answer at the back of the book.

7. A meal is one of 2 soups, one of 5 mains and one of 3 fruits. How many different meals are there?

   Answer at the back of the book.

8. In how many ways can 5 different books be arranged in a row on a shelf?

   Answer at the back of the book.

KEEP IT

Copy this into your exercise book.

Product rule: *a* ways then b ways make *a* × b ways. Nothing used twice: the choices fall by one each time. 4 letters: 4 × 3 × 2 × 1 = 24.

Draw: A row of four boxes holding 4, 3, 2 and 1, with times signs between and 24 at the end.

Key words, A to Z

Every key word in this book, what it means, and the chapter that teaches it.

AND rule
P(A and B) = P(A) × P(B), for independent events. M4.6.3

arrangement
a way of putting things in order, such as the letters ABCD in a row. M4.6.8

branch
one line of a tree diagram, marked with a probability. M4.6.4

conditional probability
the chance of one event when you already know that another has happened. M4.6.7

dependent
events where one happening changes the chance of the other. M4.6.5

equally likely
having the same chance of happening, like the six faces of a fair dice. M4.6.1

event
the result you are interested in. It is made of one or more outcomes. M4.6.1

expected number
the probability times the number of trials. It is an average, not a promise. M4.6.2

given
the word that names what you already know. P(B given A) is the chance of B when A has happened. M4.6.7

independent
events where one happening does not change the chance of the other. M4.6.3

intersection
everything in both A and B. It is written A ∩ B. M4.6.6

mutually exclusive
events that cannot both happen at once, such as red and blue on one spin. M4.6.3

OR rule
P(A or B) = P(A) + P(B), for events that cannot both happen. M4.6.3

outcome
one possible result, such as a 5 on a dice or a tail on a coin. M4.6.1

probability
a number from 0 to 1 that measures how likely an event is. M4.6.1

product rule
if one choice has a ways and another has b ways, both together have a × b ways. M4.6.8

relative frequency
the number of times an event happened divided by the number of trials. It estimates the probability. M4.6.2

set
a collection of things, such as the pupils who like football. M4.6.6

tree diagram
a diagram of branches that shows the outcomes of events in order, with a chance on each branch. M4.6.4

trial
one go of an experiment, such as one toss of a coin. M4.6.2

union
everything in A or B or both. It is written A ∪ B. M4.6.6

Venn diagram
a diagram of circles in a box that shows how sets overlap. M4.6.6

without putting back
taking a counter and keeping it out, so the next pick is from a smaller bag. M4.6.5

Answers

M4.6.1. Listing all the outcomes

1. **8**

   The coin has 2 outcomes and the spinner has 4. Each coin outcome goes with each spinner outcome, so there are 2 × 4 = 8.

   - **If you chose A:** You added 2 and 4. Each coin result goes with each spinner result, so multiply: 2 × 4. Go to M4.6.1.
   - **If you chose C:** You counted only the spinner. The coin has 2 outcomes too, so every spinner result comes twice. Go to M4.6.1.

2. **1/12**

   There are 12 equally likely outcomes. Only T5 is a tail and a 5, so the probability is 1 out of 12.

   - **If you chose A:** You used only the dice. The coin must also show a tail, which is half of the cases. Go to M4.6.1.
   - **If you chose B:** You used only the coin. The dice must also show a 5, and that is 1 case in 6. Go to M4.6.1.

3. **5/36**

   A total of 8 is 2 and 6, 3 and 5, 4 and 4, 5 and 3, 6 and 2. That is 5 cells of 36.

   - **If you chose B:** You counted the 11 totals as if they were equally likely. They are not. Count the cells of the 36 pairs. Go to M4.6.1.
   - **If you chose C:** You used the total, 8, as the number of favourable outcomes. Count the hatched cells instead. Go to M4.6.1.

4. **HH, HT, TH, TT**

   The first coin can be H or T, and with each the second coin can be H or T. That gives 4 outcomes.

   - **If you chose A:** These are only two of the outcomes. A head with a tail can happen in two ways. Go to M4.6.1.
   - **If you chose B:** HT and TH are different outcomes. In HT the first coin is a head, and in TH the second coin is a head. Go to M4.6.1.
   - **If you chose C:** This lists one coin. Two coins make pairs, one result from each. Go to M4.6.1.

5. **1/2**

   The 4 outcomes are HH, HT, TH and TT, all equally likely. HT and TH have one of each, so the probability is 2 out of 4, which is 1/2.

   - **If you chose A:** You listed HH, HT and TT. HT and TH are different outcomes, so there are 4 outcomes and 2 of them have one of each. Go to M4.6.1.
   - **If you chose C:** That is the chance of one particular outcome, such as HT. One head and one tail can happen two ways, HT and TH. Go to M4.6.1.

6. **6 times as likely**

   6 cells for a total of 7 and 1 cell for a total of 12. 6 ÷ 1 = 6, so a total of 7 is 6 times as likely.

   - **If you chose A:** You used the total, 12. Compare the numbers of ways: 6 ways against 1 way. Go to M4.6.1.
   - **If you chose B:** You took away, 6 − 1. "Times as likely" compares by dividing. Go to M4.6.1.

7. **1/9**

   A total of 9 is 3 and 6, 4 and 5, 5 and 4, 6 and 3. That is 4 of 36 cells, and 4/36 = 1/9.

8. **1/6**

   The outcomes are T5 and T6. That is 2 of the 12 equally likely outcomes, and 2/12 = 1/6.

M4.6.2. Relative frequency and expected results

1. **0.3**

   Relative frequency = 30 ÷ 100 = 0.3.

   - **If you chose A:** That is the relative frequency of landing the other way up, 70 out of 100. Go to M4.6.2.
   - **If you chose B:** That is the number of times, not a fraction of the drops. Divide it by the 100 drops. Go to M4.6.2.

2. **Drop it as many times as you can, and divide.**

   More trials usually give an estimate nearer the true probability. The estimates for 10 or 20 drops swing more than the estimate for 500.

   - **If you chose B:** A feeling is not an experiment, and a few drops are too few to settle near the true value. Go to M4.6.2.
   - **If you chose C:** Stopping when the number looks right chooses the answer you wanted. Decide the number of trials first. Go to M4.6.2.

3. **50**

   Expected number = 1/4 × 200 = 50.

   - **If you chose A:** You used the bottom of the fraction. Expected number = probability × trials = 1/4 × 200. Go to M4.6.2.
   - **If you chose B:** That is the number to expect for not green, 3/4 of 200. Go to M4.6.2.
   - **If you chose C:** You divided 200 by 1/4. Multiply the probability by the number of trials instead. Go to M4.6.2.

4. **It is close to the 100 heads that a fair coin is expected to give, so a small difference is normal.**

   For a fair coin the expected number is 1/2 × 200 = 100. It is an average over many sets of tosses, so real results are often a little off. 106 is a small difference.

   - **If you chose A:** A fair coin rarely gives exactly half. The difference here is small, and chance explains it. Go to M4.6.2.
   - **If you chose C:** Coins do not make up for earlier tosses. Each toss of a fair coin is a new chance of 1/2. Go to M4.6.2.

5. **It is an estimate that usually gets closer to the probability as the trials increase.**

   Relative frequency estimates the probability. More trials usually give a better estimate, but never a proof.

   - **If you chose A:** It is only an estimate, and 100 trials still leave room for chance. The coin gave 0.52 after 100 tosses. Go to M4.6.2.
   - **If you chose B:** It is near 0.5 for a fair coin with many tosses. It is rarely exactly 0.5. Go to M4.6.2.

6. **1/12**

   A total of 4 is 1 and 3, 2 and 2, 3 and 1. That is 3 of 36 equally likely pairs, and 3/36 = 1/12.

7. **0.18**

   Relative frequency = 45 ÷ 250 = 0.18.

8. **90**

   Expected number = 3/5 × 150 = 90.

M4.6.3. The OR rule and the AND rule

1. **3/10**

   The events cannot both happen, so add: 2/10 + 1/10 = 3/10.

   - **If you chose B:** You multiplied 2/10 × 1/10. "Or" for events that cannot both happen means add. Go to M4.6.3.
   - **If you chose C:** You took away 2/10 − 1/10. One spin cannot land on both, so you add the two chances. Go to M4.6.3.

2. **A score of one, or a score of six.**

   One roll cannot be a one and a six at once, so the events cannot both happen. Then P(one or six) = 1/6 + 1/6.

   - **If you chose A:** A 4 and a 6 are in both events. Adding counts them twice. Go to M4.6.3.
   - **If you chose B:** A 3 and a 4 are in both events, so the events can both happen. Go to M4.6.3.
   - **If you chose C:** A 3 and a 5 are in both events. Prime scores are 2, 3 and 5, and odd scores are 1, 3 and 5. Go to M4.6.3.

3. **1/20**

   The events are independent, so multiply: 1/5 × 1/4 = 1/20.

   - **If you chose A:** You added 1/5 + 1/4. For "and" with independent events, multiply. Go to M4.6.3.
   - **If you chose C:** Both must be late, so the chance must be smaller than either chance on its own. Go to M4.6.3.

4. **1/2, because each toss is independent of the ones before.**

   Each toss is a new chance. The earlier tosses cannot change a fair coin, so the chance stays 1/2.

   - **If you chose A:** The coin has no memory. Past tosses do not change the next one. Go to M4.6.3.
   - **If you chose B:** A streak is chance. It does not change the next toss of a fair coin. Go to M4.6.3.

5. **One happening does not change the chance of the other.**

   That is what independent means. Then you may multiply the chances to find the chance of both.

   - **If you chose B:** That describes events that cannot both happen, which are mutually exclusive. Independent events can happen together, like a head and a six. Go to M4.6.3.
   - **If you chose C:** A head has chance 1/2 and a six has chance 1/6, and they are independent. The chances need not match. Go to M4.6.3.

6. **120**

   Expected number = probability × trials = 2/5 × 300 = 120.

7. **2/5**

   Independent, so multiply: 2/3 × 3/5 = 6/15, and 6/15 = 2/5.

8. **1/2**

   The events cannot both happen, so add: 3/10 + 2/10 = 5/10, which is 1/2.

M4.6.4. Tree diagrams

1. **1/2, 1/3 and 1/6**

   The branches from one point must add to 1. 1/2 + 1/3 + 1/6 = 3/6 + 2/6 + 1/6 = 1.

   - **If you chose A:** These add to 5/6. An outcome with chance 1/6 is missing, so the branches do not cover everything. Go to M4.6.4.
   - **If you chose B:** These add to 3/4. A quarter of the chance is missing. Go to M4.6.4.
   - **If you chose C:** These add to 6/5, which is more than 1. The chances of the branches from one point share out exactly 1. Go to M4.6.4.

2. **4/25**

   The counter is put back, so each pick has chance 2/5 of red. 2/5 × 2/5 = 4/25.

   - **If you chose A:** That is the chance of red on one pick. Both picks must be red, so multiply the two chances. Go to M4.6.4.
   - **If you chose C:** You added 2/5 + 2/5. A path is "and then", so multiply along it. Go to M4.6.4.

3. **3/8**

   Three paths have exactly one head. Each has chance 1/8, so the total is 3 × 1/8 = 3/8.

   - **If you chose A:** That is the chance of one path, such as HTT. Three paths have exactly one head: HTT, THT and TTH. Go to M4.6.4.
   - **If you chose B:** That is the chance of a head on one toss. Count the paths with exactly one head and add their chances. Go to M4.6.4.

4. **Multiply the branches: 9/25.**

   The path is "red and then red". Multiply: 3/5 × 3/5 = 9/25.

   - **If you chose B:** A chance of 6/5 is more than 1, which is impossible. Along a path each branch narrows the chance, so multiply. Go to M4.6.4.
   - **If you chose C:** That is the chance of red on one pick. Both picks must be red, so multiply. Go to M4.6.4.

5. **4/9**

   Win then lose is 2/3 × 1/3 = 2/9. Lose then win is 1/3 × 2/3 = 2/9. Add the two paths: 4/9.

   - **If you chose A:** That is only one path, win then lose. Lose then win is a second path with the same chance, so add them. Go to M4.6.4.
   - **If you chose B:** That is the chance of winning one game. For two games, multiply along the branches and add the paths. Go to M4.6.4.
   - **If you chose C:** That is the chance of losing one game. Go to M4.6.4.

6. **1/12**

   Independent, so multiply: 1/4 × 1/3 = 1/12.

7. **9/16**

   P(red) = 3/4 each time. 3/4 × 3/4 = 9/16.

8. **3/8**

   Win then lose is 3/4 × 1/4 = 3/16. Lose then win is also 3/16. Add them: 6/16 = 3/8.

M4.6.5. Picking without putting back

1. **3/5**

   After a red is kept, 3 red and 2 blue are left. That is 5 counters, so the chance of red is 3/5.

   - **If you chose A:** You used the bag as it was at the start. A red counter has gone, so the numbers change. Go to M4.6.5.
   - **If you chose C:** You took 1 off the red counters but forgot that the bag has 1 fewer counter in all. Go to M4.6.5.

2. **2/5**

   P(both red) = 4/6 × 3/5 = 12/30 = 2/5.

   - **If you chose A:** You used 4/6 twice, as if the counter were put back. After a red is kept, the bag has 3 red in 5. Go to M4.6.5.
   - **If you chose B:** That is the chance for the first pick only. The second counter must be red too. Go to M4.6.5.

3. **3/4**

   The blue counter is kept. Now 6 red and 2 blue are left, 8 in all, so the chance of red is 6/8 = 3/4.

   - **If you chose B:** That is the chance for the first pick. A blue counter has gone, so there are 8 left. Go to M4.6.5.
   - **If you chose C:** You took a red counter away, but the counter taken was blue. All 6 red are still in the bag. Go to M4.6.5.

4. **6/42**

   The ends of a tree add to 1, which is 42/42. 12 + 12 + 12 = 36, so the missing end is 42 − 36 = 6, which is 6/42.

   - **If you chose A:** Check by adding: 12 + 12 + 12 + 9 = 45, which is more than 42. The ends must add to 42/42. Go to M4.6.5.
   - **If you chose C:** Check by adding: 12 + 12 + 12 + 4 = 40, which is less than 42. The ends must add to 42/42. Go to M4.6.5.

5. **2/15**

   P(both blue) = 4/10 × 3/9 = 12/90 = 2/15.

   - **If you chose A:** You used 2/5 twice, as if the counter were put back. After a blue is kept, the bag has 3 blue in 9. Go to M4.6.5.
   - **If you chose B:** That is the chance for the first pick alone. The second counter must be blue too. Go to M4.6.5.
   - **If you chose C:** That is the chance of blue on the second pick only, 3/9. Multiply by the chance of blue on the first pick, 4/10. Go to M4.6.5.

6. **4/9**

   Putting back keeps the chances the same: P(red) = 2/3 each time. 2/3 × 2/3 = 4/9.

7. **2/7**

   P(both red) = 4/7 × 3/6 = 12/42 = 2/7.

8. **3/5**

   Red then blue: 3/5 × 2/4 = 6/20. Blue then red: 2/5 × 3/4 = 6/20. Add: 12/20 = 3/5.

M4.6.6. Venn diagrams and sets

1. **29/40**

   Add the three regions that touch a circle: 14 + 8 + 7 = 29. Or 22 + 15 − 8 = 29. The chance is 29 out of the 40 in the box.

   - **If you chose A:** You added 22 + 15, the totals of the two circles. That counts the 8 in both twice. Add the three regions instead. Go to M4.6.6.
   - **If you chose B:** That is only the learners in music. Learners in art only are in the union too. Go to M4.6.6.
   - **If you chose D:** You divided by 22 + 15. The chance is out of all 40 learners in the box. Go to M4.6.6.

2. **1/5**

   8 of the 40 learners are in both, so the probability is 8/40 = 1/5.

   - **If you chose B:** You divided by the learners in music. The chance is out of all 40 learners in the box. Go to M4.6.6.
   - **If you chose C:** You divided by the learners in music or art. The whole group is the 40 in the box. Go to M4.6.6.

3. **Everything in both A and B.**

   A ∩ B, the intersection, is the overlap of the two circles. Its chance is the chance of A and B.

   - **If you chose A:** That is the union, A ∪ B. The intersection is smaller: only what is in both. Go to M4.6.6.
   - **If you chose C:** That is A′, written with a small mark after the letter. The sign ∩ joins two sets. Go to M4.6.6.

4. **8**

   The 14 who play chess include the 6 who play both. The chess only region holds 14 − 6 = 8.

   - **If you chose A:** That is everyone who plays chess. It includes the 6 who play both, and they sit in the overlap, not in the chess only region. Go to M4.6.6.
   - **If you chose B:** You added the 6 to the 14. The 14 already includes those 6, so take them off. Go to M4.6.6.
   - **If you chose C:** That is the overlap, the learners who play both. Chess only is the rest of the chess circle. Go to M4.6.6.

5. **3/10**

   P(A′) = 1 − P(A) = 10/10 − 7/10 = 3/10.

   - **If you chose A:** That is P(A) again. A′ means not A, which is what is left. Go to M4.6.6.
   - **If you chose B:** That compares 3 with 7. A chance is out of the whole, which is 10. Go to M4.6.6.

6. **3/10**

   P(both red) = 3/5 × 2/4 = 6/20 = 3/10.

7. **6**

   Football or netball: 25 + 18 − 9 = 34. Neither: 40 − 34 = 6.

8. **0.8**

   P(A ∪ B) = 0.6 + 0.3 − 0.1 = 0.8.

M4.6.7. Conditional probability

1. **3/4**

   Look at the Staff row only. It has 18 + 6 = 24 people, and 18 have tea. 18/24 = 3/4.

   - **If you chose B:** You divided by all 50 people. You know the person is staff, so only the 24 staff count. Go to M4.6.7.
   - **If you chose C:** That is the chance of tea for anyone in the group. You know the person is staff, so look only at the Staff row. Go to M4.6.7.

2. **9/11**

   Look at the Tea column only. It has 18 + 4 = 22 people, and 18 are staff. 18/22 = 9/11.

   - **If you chose A:** That is P(tea given staff). Here you know the person has tea, so look at the Tea column. Go to M4.6.7.
   - **If you chose C:** You divided by all 50 people. You know the person has tea, so only the 22 tea drinkers count. Go to M4.6.7.

3. **The chance of A when you know that B has happened.**

   The word "given" names what you already know. You work in the world where B has happened.

   - **If you chose A:** That is P(A and B). In P(A given B), B is already known to have happened, so it is not a chance any more. Go to M4.6.7.
   - **If you chose B:** The order of events is not the point. What matters is that B is known to have happened. Go to M4.6.7.

4. **1/2**

   P(B given A) = P(A and B) ÷ P(A) = 1/5 ÷ 2/5 = 1/5 × 5/2 = 1/2.

   - **If you chose A:** That is P(A and B), out of everyone. Divide by P(A) to look only at A. Go to M4.6.7.
   - **If you chose B:** That is P(A). To find "B given A", divide P(A and B) by it. Go to M4.6.7.
   - **If you chose C:** You multiplied 1/5 × 2/5. To find "given", divide by P(A). Go to M4.6.7.

5. **Being staff and having tea are not independent, because knowing the person is staff changes the chance of tea.**

   If the events were independent, P(tea given staff) would equal P(tea). They differ, so knowing "staff" changes the chance of tea.

   - **If you chose B:** Independent means the two chances are equal. 3/4 and 11/25 are not equal. Go to M4.6.7.
   - **If you chose C:** They match only when the events are independent. Here they differ, so the events are not independent. Go to M4.6.7.

6. **7**

   Football or netball: 30 + 25 − 12 = 43. Neither: 50 − 43 = 7.

7. **3/8**

   You know the learner is in the choir, so only the 24 in the choir count. 9 of them play the drum. 9/24 = 3/8.

8. **1/2**

   P(B given A) = P(A and B) ÷ P(A) = 3/20 ÷ 6/20 = 3/6 = 1/2.

M4.6.8. Counting choices

1. **20**

   There are 4 ways to choose the book and 5 ways to choose the pen. 4 × 5 = 20.

   - **If you chose A:** You added 4 + 5. Each book can go with each pen, so multiply. Go to M4.6.8.
   - **If you chose B:** You used 4 twice, 4 × 4. The second choice has 5 pens, not 4. Go to M4.6.8.
   - **If you chose C:** You used 5 twice, 5 × 5. The first choice has 4 books. Go to M4.6.8.

2. **125**

   Each wheel has 5 ways. 5 × 5 × 5 = 125.

   - **If you chose A:** You added 5 + 5 + 5. The choice on each wheel combines with every choice on the others, so multiply. Go to M4.6.8.
   - **If you chose B:** You took 5 × 4 × 3, as if a digit could not repeat. A wheel may show the same digit as another wheel. Go to M4.6.8.

3. **6**

   There are 3 ways to choose the first person, 2 for the second and 1 for the last. 3 × 2 × 1 = 6.

   - **If you chose A:** That counts the people, not the ways to line them up. Call them A, B and C, and list ABC, ACB, BAC and so on. Go to M4.6.8.
   - **If you chose C:** You used 3 choices for every place. A person can stand in only one place, so it is 3 × 2 × 1. Go to M4.6.8.

4. **60**

   First place can go to 5 runners, second to 4 who are left, and third to 3. 5 × 4 × 3 = 60.

   - **If you chose B:** You multiplied 5 runners by 3 places. Each place is a new choice with one runner fewer, so multiply 5 × 4 × 3. Go to M4.6.8.
   - **If you chose C:** A runner cannot win two places. The second place has only 4 choices left, and the third has 3. Go to M4.6.8.

5. **1/12**

   There are 4 × 3 × 2 × 1 = 24 equally likely arrangements. If A and B come first, C and D can follow in 2 ways. So 2 of the 24 start with AB, and 2/24 = 1/12.

   - **If you chose A:** That counts only ABCD. ABDC also starts with AB, so 2 of the 24 arrangements fit. Go to M4.6.8.
   - **If you chose C:** You used 4 × 4. A letter cannot repeat, so there are 4 × 3 × 2 × 1 = 24 arrangements, and 2 of them start with AB. Go to M4.6.8.
   - **If you chose D:** That is the chance that A comes first. The second place must be B too. Go to M4.6.8.

6. **2/5**

   P(B given A) = P(A and B) ÷ P(A) = 4/20 ÷ 10/20 = 4/10 = 2/5.

7. **30**

   Multiply the ways for each choice: 2 × 5 × 3 = 30.

8. **120**

   There are 5 ways to choose the first book, then 4, 3, 2 and 1. 5 × 4 × 3 × 2 × 1 = 120.

About this book

For learners who can find the probability of an event when the outcomes are equally likely, such as rolling an even number on a dice. In many schools this fits teenagers of about 14 to 16, and adults who want to refresh chance, trees and sets can start here too. Each chapter takes one or two sessions. A learner can work alone: every exercise has its answer and the reason for it.

Written and drawn by attnom. Edition 1.0, 2026-11-12.

This book is free. You may read it, print it, copy it and share it with anyone.

The newest copy is at https://attnom.com/academy/probability-of-events.